A composable language for action models

Tim French James Hales [1] Edwin Tay

Computer Science and Software Engineering
The University of Western Australia
Perth, Australia

Abstract

Action models are semantic structures similar to Kripke models that represent a change in knowledge in an epistemic setting. Whereas the language of action model logic [8,7] embeds the semantic structure of an action model directly within the language, this paper introduces a language that represents action models using syntactic operators inspired by relational actions [11,12,13]. This language admits an intuitive description of the action models it represents, and we show in several settings that it is sufficient to represent any action model up to a given modal depth and to represent the results of action model synthesis [19], and give a sound and complete axiomatisation in some of these settings.

Keywords: Modal logic, Epistemic logic, Doxastic logic, Temporal epistemic logic, Multi-agent system, Action model logic

1 Introduction

Dynamic epistemic logic describes the way knowledge can change in multi-agent systems subject to informative actions taking place. For example, if Tim were to announce "I like cats", then everyone in the room would know the proposition *Tim likes cats* is true, and furthermore, everybody would know that this fact is common knowledge among the people in the room. This simple informative action is what is referred to as a public announcement [23], and such actions of these have been extensively studied in epistemic logics. More complex actions can include private announcements (where some agents are oblivious to the informative action occurring), or a group announcement (where members of a group simultaneously make a truthful announcement to every other member of the group [1]). These complex actions may be modelled and reasoned about using action models [8] which are effectively a semantic model of the change caused by an informative action. Consequently they are very useful for reasoning about the consequences of an informative action, but less well suited to reasoning about the action itself.

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We present a language for describing epistemic actions syntactically. Complex actions may be built as an expression upon simpler primitive actions. This approach is a generalisation of the relational actions introduced by van Ditmarsch [12]. We show in several settings that this language is sufficient to represent any informative action represented by an action model (up to a given model depth), we present a synthesis result, and give a a sound and complete axiomatisation for some of these settings. The synthesis result is an important application of this work: given a desired state of knowledge among a group of agents, we are able to compute a complex informative action that will achieve that particular knowledge state (given it is consistent with the current knowledge of agents). We provided these results in a variety of modal logics suited to epistemic reasoning: $\mathcal{K}$, $\mathcal{K}45$ and $\mathcal{S}5$.

Example 1.1 James, Ed and Tim submit a research grant proposal, and eagerly await the outcome. Is there a series of actions that will result in:

(i) Ed knowing the grant application was successful;

(ii) James not knowing whether the grant application was successful, but knowing that either Ed or Tim does know;

(iii) Tim does not know whether the grant application was successful, but knows that if the grant application was unsuccessful, then James knows that it was unsuccessful.

Such an epistemic state may be achieved by a series of messages: Ed is sent a message congratulating him on a successful application, James is sent a message informing him that at least one applicant on each grant has been informed of the outcome, and Tim is sent a message informing him that the first investigator of all unsuccessful grants has been notified.

After establishing some technical preliminaries (Section 2) we present a syntactic approach for describing informative actions (Sections 3 and 4), provide a sound and complete axiomatisation of the language (Section 5) and provide a correspondence result between this language and action models (Section 6), give a computational method for synthesising actions to achieve an epistemic goal (Section 7).

2 Technical Preliminaries

We recall definitions from modal logic, the action model logic of Baltag, Moss and Solecki [8,7] the refinement modal logic of van Ditmarsch, French and Pinchinat [14,15] and the arbitrary action model logic of Hales [19].

Let P be a non-empty, countable set of propositional atoms, and let A be a non-empty, finite set of agents.

Definition 2.1 [Kripke model] A *Kripke model* $M = (S, R, V)$ consists of a *domain* S, which is a non-empty set of states (or possible worlds), an *accessibility* function $R : A \to \mathcal{P}(S \times S)$, which is a function from agents to accessibility relations on S, and a *valuation* function $V : P \to \mathcal{P}(S)$, which is a function from states to sets of propositional atoms.

The *class of all Kripke models* is called $\mathcal{K}$. A *multi-pointed Kripke model* $M_T = (M, T)$ consists of a Kripke model M along with a designated set of states $T \subseteq S$.

We write R_a to denote $R(a)$. Given two states $s, t \in S$, we write sR_at to denote that $(s,t) \in R_a$. We write TR_a to denote the set of states $\{s \in S \mid t \in T, tR_as\}$ and write R_aT to denote the set of states $\{s \in S \mid t \in T, sR_at\}$. We write M_s as an abbreviation for $M_{\{s\}}$, and write tR_a and R_at as abbreviations for $\{t\}R_a$ and $R_a\{t\}$ respectively. As we will often be required to discuss several models at once, we will use the convention that $M_T = ((S, R, V), T)$, $M'_{T'} = ((S', R', V'), T')$, $M^\gamma_{T^\gamma} = ((S^\gamma, R^\gamma, V^\gamma), T^\gamma)$, etc.

Definition 2.2 [Action model] Let $\mathcal{L}$ be a logical language. An *action model* $\mathsf{M} = (\mathsf{S}, \mathsf{R}, \mathsf{pre})$ with preconditions defined on $\mathcal{L}$ consists of a *domain* S, which is a non-empty, finite set of action points, an *accessibility* function $\mathsf{R} : A \to \mathcal{P}(\mathsf{S} \times \mathsf{S})$, which is a function from agents to accessibility relations on S, and a *precondition* function $\mathsf{pre} : \mathsf{S} \to \mathcal{L}$, which is a function from action points to formulae from $\mathcal{L}$.

The *class of all action models* is called $\mathcal{AM}$. A *multi-pointed action model* $\mathsf{M}_\mathsf{T} = (\mathsf{M}, \mathsf{T})$ consists of an action model M along with a designated set of action points $\mathsf{T} \subseteq \mathsf{S}$.

We use the same abbreviations and conventions for action models as are used for Kripke models. We use the convention of using sans-serif fonts for action models, as in M_T and italic fonts for Kripke models, as in M_T.

In addition to the class $\mathcal{K}$ of all Kripke models, and the class $\mathcal{AM}$ of all action models we will be referring to several other classes of Kripke models and action models.

Definition 2.3 [Classes of Kripke models and action models] The class of all Kripke models / action models with transitive and Euclidean accessibility relations is called $\mathcal{K}45$ / $\mathcal{AM}_{\mathcal{K}45}$.

The class of all Kripke models / action models with serial, transitive and Euclidean accessibility relations is called $\mathcal{KD}45$ / $\mathcal{AM}_{\mathcal{KD}45}$.

The class of all Kripke models / action models with reflexive, transitive and Euclidean accessibility relations is called $\mathcal{S}5$ / $\mathcal{AM}_{\mathcal{S}5}$.

Definition 2.4 [Language of arbitrary action model logic] The language $\mathcal{L}_{\otimes\forall}$ of arbitrary action model logic is inductively defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\mathsf{M}_\mathsf{T}]\varphi \mid \forall\varphi$$

where $p \in P$, $a \in A$, and $\mathsf{M}_\mathsf{T} \in \mathcal{AM}$ is a multi-pointed action model with preconditions defined on the language $\mathcal{L}_{\otimes\forall}$.

We use all of the standard abbreviations for propositional logic, in addition to the abbreviations $\Diamond_a\varphi ::= \neg\Box_a\neg\varphi$, $\langle\mathsf{M}_\mathsf{T}\rangle\varphi ::= \neg[\mathsf{M}_\mathsf{T}]\neg\varphi$, and $\exists\varphi ::= \neg\forall\neg\varphi$.

We also use the cover operator of Janin and Walukiewicz [22], following the definitions given by Bílková, Palmigiano and Venema [9]. The cover operator, $\nabla_a\Gamma$ is an abbreviation defined by $\nabla_a\Gamma ::= \Box_a \bigvee_{\gamma\in\Gamma} \gamma \wedge \bigwedge_{\gamma\in\Gamma} \Diamond_a\gamma$, where $\Gamma \subseteq$

$\mathcal{L}_{\otimes\forall}$ is a finite set of formulae. We note that the modal operators $\Box_a$, $\Diamond_a$ and ∇_a are interdefineable as $\Box_a\varphi \leftrightarrow \nabla_a\{\varphi\} \vee \nabla_a\emptyset$ and $\Diamond_a\varphi \leftrightarrow \nabla_a\{\varphi, \top\}$. This is the basis for the axiomatisations of refinement modal logic and arbitrary action model logic, and plays an important part in our correspondence and synthesis results. This was previously used as the basis of several axiomatisations of refinement modal logics [15,20,18,21,10,19].

We refer to the language $\mathcal{L}_{\otimes}$ of action model logic, which is $\mathcal{L}_{\otimes\forall}$ without the $\forall$ operator, the language $\mathcal{L}_{\forall}$ of refinement modal logic, which is $\mathcal{L}_{\otimes\forall}$ without the $[\mathsf{M}_\mathsf{T}]$ operator, the language $\mathcal{L}$ of modal logic, which is $\mathcal{L}_{\otimes}$ without the $[\mathsf{M}_\mathsf{T}]$ operator, and the language $\mathcal{L}_0$ of propositional logic, which is $\mathcal{L}$ without the $\Box_a$ operator.

Definition 2.5 [Semantics of modal logic] Let $\mathcal{C}$ be a class of Kripke models and let $M = (S, R, V) \in \mathcal{C}$ be a Kripke model. The interpretation of $\varphi \in \mathcal{L}$ in the logic C is defined inductively as:

$$\begin{aligned}
M_s \models p &\text{ iff } s \subseteq V(p)\\
M_s \models \neg\varphi &\text{ iff } M_s \not\models \varphi\\
M_s \models \varphi \wedge \psi &\text{ iff } M_s \models \varphi \text{ and } M_s \models \psi\\
M_s \models \Box_a\varphi &\text{ iff for every } t \in sR_a : M_t \models \varphi\\
M_T \models \varphi &\text{ iff for every } t \in T : M_t \models \varphi
\end{aligned}$$

Definition 2.6 [Bisimilarity of Kripke models] Let $M = (S, R, V) \in \mathcal{K}$ and $M' = (S', R', V') \in \mathcal{K}$ be Kripke models. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a *bisimulation* if and only if for every $a \in A$ and $(s, s') \in \mathfrak{R}$ the following conditions hold:

atoms For every $p \in P$: $s \in V(p)$ if and only if $s' \in V'(p)$.

forth-a For every $t \in sR_a$ there exists $t' \in s'R'_a$ such that $(t, t') \in \mathfrak{R}$.

back-a For every $t' \in s'R'_a$ there exists $t \in sR_a$ such that $(t, t') \in \mathfrak{R}$.

If $(s, s') \in \mathfrak{R}$ then we call M_s and $M'_{s'}$ *bisimilar* and write $M_s \leftrightarroweq M'_{s'}$.

Proposition 2.7 *The relation* $\leftrightarroweq$ *is an equivalence relation on Kripke models.*

Proposition 2.8 *Let* $M_s, M'_{s'} \in \mathcal{K}$ *be Kripke models such that* $M_s \leftrightarroweq M'_{s'}$*. Then for every* $\varphi \in \mathcal{L}$*:* $M_s \models \varphi$ *if and only if* $M'_{s'} \models \varphi$*.*

These are well-known results.

Definition 2.9 [n-bisimilarity of Kripke models] Let $n \in \mathbb{N}$, and let $M_s = ((S, R, V), s) \in \mathcal{K}$ and $M'_{s'} = ((S', R', V'), s') \in \mathcal{K}$ be Kripke models. We say that M_s is *n-bisimilar* to $M'_{s'}$, and write $M_s \leftrightarroweq_n M'_{s'}$, if and only if for every $a \in A$ the following conditions hold:

atoms For every $p \in P$: $s \in V(p)$ if and only if $s' \in V'(p)$.

forth-n-a If $n > 0$ then for every $t \in sR_a$ there exists $t' \in s'R'_a$ such that $M_t \leftrightarroweq_{(n-1)} M'_{t'}$

back-n-a If $n > 0$ then for every $t' \in s'R'_a$ there exists $t \in sR_a$ such that $M_t \leftrightarroweq_{(n-1)} M'_{t'}$

Definition 2.10 [Modal depth] Let $\varphi \in \mathcal{L}$. The *modal depth of* φ, written as $d(\varphi)$, is defined recursively as follows:

$$\begin{aligned} d(p) &= 0 \text{ for } p \in P \\ d(\neg\psi) &= d(\psi) \\ d(\psi \wedge \chi) &= max(d(\psi), d(\chi)) \\ d(\Box_a \psi) &= 1 + d(\psi) \end{aligned}$$

Proposition 2.11 *The relation* $\leftrightarroweq_n$ *is an equivalence relation on Kripke models.*

Proposition 2.12 *Let* $n \in \mathbb{N}$ *and let* $M_s, M'_{s'} \in \mathcal{K}$ *be Kripke models such that* $M_s \leftrightarroweq_n M'_{s'}$*. If* $m < n$ *then* $M_s \leftrightarroweq_m M'_{s'}$*.*

Proposition 2.13 *Let* $n \in \mathbb{N}$ *and let* $M_s, M'_{s'} \in \mathcal{K}$ *be Kripke models such that* $M_s \leftrightarroweq_n M'_{s'}$*. Then for every* $\varphi \in \mathcal{L}$ *such that* $d(\varphi) \leq n$*:* $M_s \models \varphi$ *if and only if* $M'_{s'} \models \varphi$*.*

Proposition 2.14 *Let* $M_s, M'_{s'} \in \mathcal{K}$ *be Kripke models. Then* $M_s \leftrightarroweq M'_{s'}$ *if and only if for every* $n \in \mathbb{N}$*:* $M_s \leftrightarroweq_n M'_{s'}$*.*

These are well-known results.

Definition 2.15 [B-bisimilarity of Kripke models] Let $M_s = ((S, R, V), s) \in \mathcal{K}$ and $M'_{s'} = ((S', R', V'), s') \in \mathcal{K}$ be Kripke models. We say that M_s is *B-bisimilar* to $M'_{s'}$, and write $M_s \leftrightarroweq_B M'_{s'}$, if and only if for every $b \in B$ the following conditions hold:

atoms For every $p \in P$: $s \in V(p)$ if and only if $s' \in V'(p)$.

forth-b For every $t \in sR_b$ there exists $t' \in s'R'_b$ such that $M_t \leftrightarroweq M'_{t'}$.

back-b For every $t' \in s'R'_b$ there exists $t \in sR_b$ such that $M_t \leftrightarroweq M'_{t'}$.

Definition 2.16 [B-restricted formulae] Let $B \subseteq A$. A *B-restricted formula* is defined by the following abstract syntax:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_b \psi$$

where $p \in P$, $b \in B$, $\psi \in \mathcal{L}$.

Proposition 2.17 *Let* $B \subseteq A$*, and* $M_s, M'_{s'} \in \mathcal{K}$ *be Kripke models such that* $M_s \leftrightarroweq_B M'_{s'}$*. Then for every* $\varphi \in \mathcal{L}$ *such that* φ *is is a B-restricted formula:* $M_s \models \varphi$ *if and only if* $M'_{s'} \models \varphi$*.*

This result is trivial.

We recall the semantics of action model logic of Baltag, Moss and Solecki [8,7].

Definition 2.18 [Semantics of action model logic] Let $\mathcal{C}$ be a class of Kripke models, let $M = (S, R, V) \in \mathcal{C}$ be a Kripke model and let $\mathsf{M} \in \mathcal{AM}$ be an action model.

We first define *action model execution.* We denote the result of executing the action model M on the Kripke model M as $M \otimes \mathsf{M}$, and we define the result as $M \otimes \mathsf{M} = M' = (S', R', V')$ where:

$$S' = \{(s, \mathsf{s}) \mid s \in S, \mathsf{s} \in \mathsf{S}, M_s \models \mathsf{pre}(\mathsf{s})\}$$
$$(s, \mathsf{s})R'_a(t, \mathsf{t}) \text{ iff } sR_at \text{ and } \mathsf{s}\mathsf{R}_a\mathsf{t}$$
$$(s, \mathsf{s}) \in V'(p) \text{ iff } s \in V(p)$$

We also define *multi-pointed action model execution* as $M_T \otimes \mathsf{M}_\mathsf{T} = M'_{T'} = ((S', R', V'), T') = ((M \otimes \mathsf{M}), (T \times \mathsf{T}) \cap S')$.

Then the interpretation of $\varphi \in \mathcal{L}_\otimes$ in the logic $C_\otimes$ is the same as its interpretation in the modal logic C given in Definition 2.5, with the additional inductive case:

$$M_s \models [\mathsf{M}_\mathsf{T}]\varphi \text{ iff } M_s \otimes \mathsf{M}_\mathsf{T} \in \mathcal{C} \text{ implies } M_s \otimes \mathsf{M}_\mathsf{T} \models \varphi$$

Definition 2.19 [Sequential execution of action models] Let $\mathsf{M}, \mathsf{M}' \in \mathcal{AM}$. We define the *sequential execution of* M *and* M' as $\mathsf{M} \otimes \mathsf{M}' = \mathsf{M}'' = (\mathsf{S}'', \mathsf{R}'', \mathsf{pre}'')$ where:

$$\mathsf{S}'' = \mathsf{S} \times \mathsf{S}'$$
$$(\mathsf{s}, \mathsf{s}')R''_a(\mathsf{t}, \mathsf{t}') \text{ iff } \mathsf{s}\mathsf{R}_a\mathsf{t} \text{ and } \mathsf{s}'\mathsf{R}'_a\mathsf{t}'$$
$$\mathsf{pre}''((\mathsf{s}, \mathsf{s}')) = \langle \mathsf{M}_\mathsf{s} \rangle \mathsf{pre}'(\mathsf{s}')$$

We also define *sequential action of* M_T *and* $\mathsf{M}'_{\mathsf{T}'}$ as $\mathsf{M}_\mathsf{T} \otimes \mathsf{M}'_{\mathsf{T}'} = \mathsf{M}''_{\mathsf{T}''} = ((\mathsf{S}'', \mathsf{R}'', \mathsf{pre}''), \mathsf{T} \times \mathsf{T}')$.

Definition 2.20 [Bisimilarity of action models] Let $\mathsf{M} = (\mathsf{S}, \mathsf{R}, \mathsf{pre}) \in \mathcal{AM}$ and $\mathsf{M}' = (\mathsf{S}', \mathsf{R}', \mathsf{pre}') \in \mathcal{AM}$ be action models. A non-empty relation $\mathfrak{R} \subseteq \mathsf{S} \times \mathsf{S}'$ is a *bisimulation* if and only if for every $a \in A$ and $(\mathsf{s}, \mathsf{s}') \in \mathfrak{R}$ the following conditions hold:

atoms $\vdash \mathsf{pre}(\mathsf{s}) \leftrightarrow \mathsf{pre}'(\mathsf{s}')$

forth-a For every $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ there exists $\mathsf{t}' \in \mathsf{s}'\mathsf{R}'_a$ such that $(\mathsf{t}, \mathsf{t}') \in \mathfrak{R}$.

back-a For every $\mathsf{t}' \in \mathsf{s}'\mathsf{R}'_a$ there exists $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ such that $(\mathsf{t}, \mathsf{t}') \in \mathfrak{R}$.

If $(\mathsf{s}, \mathsf{s}') \in \mathfrak{R}$ then we call M_s and $\mathsf{M}'_{\mathsf{s}'}$ *bisimilar* and write $\mathsf{M}_\mathsf{s} \leftrightarroweq \mathsf{M}'_{\mathsf{s}'}$.

Proposition 2.21 *The relation* $\leftrightarroweq$ *is an equivalence relation on action models.*

Proposition 2.22 *Let* $M_s, M'_{s'} \in \mathcal{K}$ *be Kripke models such that* $M_s \leftrightarroweq M'_{s'}$. *and let* $\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'} \in \mathcal{AM}$ *be action models such that* $\mathsf{M}_\mathsf{s} \leftrightarroweq \mathsf{M}'_{\mathsf{s}'}$. *Then* $(M_s \otimes \mathsf{M}_\mathsf{s}) \leftrightarroweq (M'_{s'} \otimes \mathsf{M}'_{\mathsf{s}'})$.

Proposition 2.23 *Let* $M_s \in \mathcal{K}$ *be a Kripke model and let* $\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'} \in \mathcal{AM}$ *be action models. Then* $((M_s \otimes \mathsf{M}_\mathsf{s}) \otimes \mathsf{M}'_{\mathsf{s}'}) \leftrightarroweq (M_s \otimes (\mathsf{M}_\mathsf{s} \otimes \mathsf{M}'_{\mathsf{s}'}))$.

These results are shown by Baltag, Moss and Solecki [8,7].

Definition 2.24 [n-bisimilarity of action models] Let $n \in \mathbb{N}$, and let $\mathsf{M}_\mathsf{s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{AM}$ and $\mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}') \in \mathcal{AM}$ be action models. We say that M_s is *n-bisimilar* to $\mathsf{M}'_{\mathsf{s}'}$, and write $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \mathsf{M}'_{\mathsf{s}'}$, if and only if for every $a \in A$ the following conditions hold:

atoms $\vdash \mathsf{pre}(\mathsf{s}) \leftrightarrow \mathsf{pre}'(\mathsf{s}')$

forth-n-a If $n > 0$ then for every $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ there exists $\mathsf{t}' \in \mathsf{s}'\mathsf{R}'_a$ such that $\mathsf{M}_\mathsf{t} \leftrightarroweq_{(n-1)} \mathsf{M}'_{\mathsf{t}'}$

back-n-a If $n > 0$ then for every $\mathsf{t}' \in \mathsf{s}'\mathsf{R}'_a$ there exists $\mathsf{t} \in \mathsf{sR}_a$ such that $\mathsf{M}_\mathsf{t} \leftrightarroweq_{(n-1)} \mathsf{M}'_{\mathsf{t}'}$

Proposition 2.25 *The relation $\leftrightarroweq_n$ is an equivalence relation on action models.*

Proposition 2.26 *Let $n \in \mathbb{N}$ and let $\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'} \in \mathcal{K}$ be action models such that $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \mathsf{M}'_{\mathsf{s}'}$. If $m < n$ then $\mathsf{M}_\mathsf{s} \leftrightarroweq_m \mathsf{M}'_{\mathsf{s}'}$.*

Proposition 2.27 *Let $n \in \mathbb{N}$, let $M_s, M'_{s'} \in \mathcal{K}$ be Kripke models such that $M_s \leftrightarroweq_n M'_{s'}$. and let $\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'} \in \mathcal{K}$ be action models such that $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \mathsf{M}'_{\mathsf{s}'}$. Then $(M_s \otimes \mathsf{M}_\mathsf{s}) \leftrightarroweq_n (M'_{s'} \otimes \mathsf{M}'_{\mathsf{s}'})$.*

Proposition 2.28 *Let $\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'} \in \mathcal{K}$ be action models. Then $\mathsf{M}_\mathsf{s} \leftrightarroweq \mathsf{M}'_{\mathsf{s}'}$ if and only if for every $n \in \mathbb{N}$: $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \mathsf{M}'_{\mathsf{s}'}$.*

These results follow from similar reasoning to the results for n-bisimilarity of Kripke models.

Definition 2.29 [B-bisimilarity of action models] Let $\mathsf{M}_\mathsf{s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{K}$ and $\mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}') \in \mathcal{K}$ be Kripke models. We say that M_s is B-bisimilar to $\mathsf{M}'_{\mathsf{s}'}$ and write $\mathsf{M}_\mathsf{s} \leftrightarroweq_B \mathsf{M}'_{\mathsf{s}'}$, if and only if for every $b \in B$ the following conditions hold:

atoms For every $p \in P$: $\mathsf{s} \in V(p)$ if and only if $\mathsf{s}' \in V'(p)$.

forth-b For every $\mathsf{t} \in \mathsf{sR}_b$ there exists $\mathsf{t}' \in \mathsf{s}'\mathsf{R}'_b$ such that $\mathsf{M}_\mathsf{t} \leftrightarroweq \mathsf{M}'_{\mathsf{t}'}$.

back-b For every $\mathsf{t}' \in \mathsf{s}'\mathsf{R}'_b$ there exists $\mathsf{t} \in \mathsf{sR}_b$ such that $\mathsf{M}_\mathsf{t} \leftrightarroweq \mathsf{M}'_{\mathsf{t}'}$.

Proposition 2.30 *Let $M_s, M'_{s'} \in \mathcal{K}$ be Kripke models such that $M_s \leftrightarroweq_B M'_{s'}$. and let $\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'} \in \mathcal{K}$ be action models such that $\mathsf{M}_\mathsf{s} \leftrightarroweq_B \mathsf{M}'_{\mathsf{s}'}$. Then $(M_s \otimes \mathsf{M}_\mathsf{s}) \leftrightarroweq_B (M'_{s'} \otimes \mathsf{M}'_{\mathsf{s}'})$.*

This result follows from similar reasoning to the results for B-bisimilarity of Kripke models.

Definition 2.31 [Axiomatisation $\mathbf{AML_K}$] The axiomatisation $\mathbf{AML_K}$ is a substitution schema consisting of the rules and axioms of $\mathbf{K}$ along with the axioms:

$$\begin{aligned}
\mathbf{AS} &\vdash [\mathsf{M}_\mathsf{T} \otimes \mathsf{M}'_{\mathsf{T}'}]\varphi \leftrightarrow [\mathsf{M}'_{\mathsf{T}'}][\mathsf{M}_\mathsf{T}]\varphi \\
\mathbf{AU} &\vdash [\mathsf{M}_\mathsf{T}]\varphi \leftrightarrow \textstyle\bigwedge_{\mathsf{t} \in \mathsf{T}} [\mathsf{M}_\mathsf{t}]\varphi \\
\mathbf{AP} &\vdash [\mathsf{M}_\mathsf{t}]p \leftrightarrow (\mathsf{pre}(\mathsf{t}) \to p) \text{ for } p \in P \\
\mathbf{AN} &\vdash [\mathsf{M}_\mathsf{t}]\neg\varphi \leftrightarrow (\mathsf{pre}(\mathsf{t}) \to \neg[\mathsf{M}_\mathsf{t}]\varphi) \\
\mathbf{AC} &\vdash [\mathsf{M}_\mathsf{t}](\varphi \wedge \psi) \leftrightarrow ([\mathsf{M}_\mathsf{t}]\varphi \wedge [\mathsf{M}_\mathsf{t}]\psi) \\
\mathbf{AK} &\vdash [\mathsf{M}_\mathsf{t}]\Box_a\varphi \leftrightarrow (\mathsf{pre}(\mathsf{t}) \to \Box_a[\mathsf{M}_{\mathsf{tR}_a}]\varphi)
\end{aligned}$$

and the rule:

$$\mathbf{NecA} \text{ From } \vdash \varphi \text{ infer } \vdash [\mathsf{M}_\mathsf{T}]\varphi$$

Proposition 2.32 *The axiomatisation $\mathbf{AML_K}$ is sound and complete for the logic $K_\otimes$.*

Proposition 2.33 *The logic $K_\otimes$ is expressively equivalent to the logic K.*

These results are shown by Baltag, Moss and Solecki [8,7]. We note that the completeness and expressive equivalence results follow from the fact that $\mathbf{AML_K}$ forms a set of reduction axioms which give a provably correct translation from $\mathcal{L}_\otimes$ to $\mathcal{L}$.

We note that the same results hold for the logics $K45_\otimes$ and $S5_\otimes$ if we extend $\mathbf{AML_K}$ with the additional axioms of **K45** and **S5** and restrict the language to only include $\mathcal{AM}_{\mathcal{K}45}$ and $\mathcal{AM}_{\mathcal{S}5}$ action models respectively, given the following results.

Proposition 2.34 *$M \in \mathcal{K}$ and $\mathsf{M} \in \mathcal{AM}_{\mathcal{K}45}$ if and only if $M \otimes \mathsf{M} \in \mathcal{K}45$.*

Proposition 2.35 *$M \in \mathcal{K}$ and $\mathsf{M} \in \mathcal{AM}_{\mathcal{S}5}$ if and only if $M \otimes \mathsf{M} \in \mathcal{S}5$.*

Definition 2.36 [Simulation and refinement] Let $M, M' \in \mathcal{K}$ be Kripke models. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a *simulation* if and only if it satisfies **atoms**, **forth-**a for every $a \in A$. If $(s, s') \in \mathfrak{R}$ then we call $M'_{s'}$ a *simulation* of M_s and call M_s a *refinement* of $M'_{s'}$. We write $M'_{s'} \rightrightarrows M_s$ or equivalently $M_s \leftleftarrows M'_{s'}$.

Proposition 2.37 *The relation $\leftleftarrows$ is a preorder on Kripke models.*

Proposition 2.38 *Let $M_s \in \mathcal{K}$ and $\mathsf{M_s} \in \mathcal{AM}$. Then $M_s \otimes \mathsf{M_s} \leftleftarrows M_s$*

These results are shown by van Ditmarsch and French [14].

Definition 2.39 [Semantics of arbitrary action model logic] Let $\mathcal{C}$ be a class of Kripke models and let $M \in \mathcal{C}$ be a Kripke model. The interpretation of $\varphi \in \mathcal{L}_{\otimes\forall}$ in the logic $C_{\otimes\forall}$ is the same as its interpretation in the action model logic $C_\otimes$ given in Definition 2.18 with the additional inductive case:

$$M_s \models \forall\varphi \text{ iff for every } M'_{s'} \in \mathcal{C} \text{ such that } M'_{s'} \leftleftarrows M_s : M'_{s'} \models \varphi$$

The semantics of arbitrary action model logic are given by Hales [19], which are a combination of the semantics of action model logic of Baltag, Moss and Solecki [8,7] and the semantics of refinement modal logic of van Ditmarsch and French [14].

As noted earlier, the action model logics $K_\otimes$, $K45_\otimes$ and $S5_\otimes$ are expressively equivalent to their underlying modal logics via a provably correct translation. Similarly it was shown by Bozzelli, et al. [10] and Hales, French and Davies [21] that the refinement modal logics $K_\forall$, $KD45_\forall$ and $S5$ are expressively equivalent to their underlying modal logics, also via a provably correct translation. We note that the same result for $K45_\forall$ can be shown similarly to the result for $KD45_\forall$. In axiomatising $K_{\otimes\forall}$, Hales [19] simply noted that the rules and axioms of $\mathbf{AML_K}$ and $\mathbf{RML_K}$ are sound in $K_{\otimes\forall}$ and that the provably correct translations for $K_\otimes$ and $K_\forall$ can be simply combined to form a provably correct translation for $K_{\otimes\forall}$. We reproduce the axiomatisation for $K_{\otimes\forall}$ here, and note that the same similar reasoning to [19] gives sound and complete axiomatisations and provably correct translations for $K45_{\otimes\forall}$ and $S5_{\otimes\forall}$, which we also list here.

Definition 2.40 [Disjunctive normal form] A formula in *disjunctive normal form* is defined by the following abstract syntax:

$$\varphi :: \pi \wedge \bigwedge_{b \in B} \nabla_a \Gamma_a \mid \varphi \vee \varphi$$

where $\pi \in \mathcal{L}_0$, $B \subseteq A$ and for every $b \in B$, Γ_b is a finite set of formulae in disjunctive normal form.

Proposition 2.41 *Every formula of $\mathcal{L}$ is equivalent to a formula in disjunctive normal form under the semantics of K.*

This is shown by van Ditmarsch, French and Pinchinat [15].

Definition 2.42 [Axiomatisation $\mathbf{AAML_K}$] The axiomatisation $\mathbf{AAML_K}$ is a substitution schema consisting of the rules and axioms of $\mathbf{AML_K}$ along with the axioms:

$$\begin{aligned} \mathbf{R}\ & \forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi) \\ \mathbf{RP}\ & \forall\pi \leftrightarrow \pi \text{ where } \pi \in \mathcal{L}_0 \\ \mathbf{RK}\ & \exists\nabla_a\Gamma_a \leftrightarrow \textstyle\bigwedge_{\gamma\in\Gamma_a} \Diamond_a \exists\gamma \\ \mathbf{RDist}\ & \exists \textstyle\bigwedge_{a\in A} \nabla_a\Gamma_a \leftrightarrow \textstyle\bigwedge_{a\in A} \exists\nabla_a\Gamma_a \end{aligned}$$

and the rule:

$$\mathbf{NecR}\ \text{From} \vdash \varphi \text{ infer} \vdash \forall\varphi$$

The additional axioms for $\mathbf{AAML_K}$ are the additional axioms from $\mathbf{RML_K}$ for refinement modal logic, given by Bozzelli, et al. [10].

Proposition 2.43 *The axiomatisation $\mathbf{AAML_K}$ is sound and complete for the logic $K_{\otimes\forall}$.*

Proposition 2.44 *The logic $K_{\otimes\forall}$ is expressively equivalent to the logic K.*

These results are shown by Hales [19].

Definition 2.45 [Alternating disjunctive normal form] A formula in *a-alternating disjunctive normal form* is defined by the following abstract syntax:

$$\varphi :: \pi \wedge \bigwedge_{b \in B} \nabla_b \Gamma_b \mid \varphi \vee \varphi$$

where $\pi \in \mathcal{L}_0$, $B \subseteq A \setminus \{a\}$ and for every $b \in B$, Γ_b is a finite set of formulae in b-alternating disjunctive normal form.

A formula in *alternating disjunctive normal form* is defined by the following abstract syntax:

$$\varphi :: \pi \wedge \bigwedge_{b \in B} \nabla_a \Gamma_a \mid \varphi \vee \varphi$$

where $\pi \in \mathcal{L}_0$, $B \subseteq A$ and for every $b \in B$, Γ_b is a finite set of formulae in b-alternating disjunctive normal form.

The additional axioms for $\mathbf{AAML_{K45}}$ are adapted from the additional axioms from $\mathbf{RML_{KD45}}$ for refinement doxastic logic, given by Hales, French and Davies [21]. The axioms do not require that each Γ_a be non-empty, which is due to the lack of seriality in the setting of $\mathcal{K}45$.

Proposition 2.46 *Every formula of $\mathcal{L}$ is equivalent to a formula in alternating disjunctive normal form under the semantics of K45.*

This is shown by Hales, French and Davies [21] for *KD45*, however the same reasoning applies to *K45*.

Definition 2.47 [Axiomatisation $\mathbf{AAML_{K45}}$] The axiomatisation $\mathbf{AML_{K45}}$ is a substitution schema consisting of the rules and axioms of $\mathbf{AML_{K45}}$ along with the rules and axioms **R**, **RP** and **NecR** of $\mathbf{AAML_K}$ and the axioms:

$$\begin{array}{ll} \textbf{RK45} & \exists\nabla_a\Gamma_a \leftrightarrow \bigwedge_{\gamma\in\Gamma_a} \Diamond_a\exists\gamma \\ \textbf{RDist} & \exists\bigwedge_{a\in A}\nabla_a\Gamma_a \leftrightarrow \bigwedge_{a\in A}\exists\nabla_a\Gamma_a \end{array}$$

where for every $a \in A$, Γ_a is a finite set of a-alternating disjunctive normal formulae.

The additional axioms for $\mathbf{AAML_{K45}}$ are the additional axioms from $\mathbf{RML_{K45}}$ for refinement epistemic logic, given by Hales, French and Davies [21].

The additional axioms for $\mathbf{AAML_{K45}}$ are adapted from the additional axioms from $\mathbf{RML_{KD45}}$ for refinement modal logic, given by Hales, French and Davies [21]. The axioms do not require that each Γ_a be non-empty, which is due to the lack of seriality in the setting of $\mathcal{K}45$.

Proposition 2.48 *The axiomatisation* $\mathbf{AAML_{K45}}$ *is sound and complete for the logic* $K45_{\otimes\forall}$.

Proposition 2.49 *The logic* $K45_{\otimes\forall}$ *is expressively equivalent to the logic K45.*

These results follow from similar reasoning to the same results in $K_{\otimes\forall}$.

Definition 2.50 [Explicit formulae] Let $\pi \in \mathcal{L}_0$ be a propositional formula, let $\gamma^0 \in \mathcal{L}$ be a modal formula and for every $a \in A$ let $\Gamma_a \subseteq \mathcal{L}$ be a finite set of formulae such that $\gamma^0 \in \Gamma_a$. Let $\Psi = \{\psi \leq \gamma \mid a \in A, \gamma \in \Gamma_a\}$ be the set of subformulae of the formulae in each set Γ_a. Finally let φ be a formula of the form

$$\varphi = \pi \wedge \gamma^0 \wedge \bigwedge_{a\in A} \nabla_a\Gamma_a$$

Then φ is an *explicit formula* if and only if the following conditions hold:

(i) For every $a \in A$, $\gamma \in \Gamma_a$, $\psi \in \Psi$: either $\vdash_{\mathbf{S5}} \gamma \rightarrow \psi$ or $\vdash_{\mathbf{S5}} \gamma \rightarrow \neg\psi$.

(ii) For every $a \in A$, $\gamma \in \Gamma_a$, $\Box_a\psi \in \Psi$: $\vdash_{\mathbf{S5}} \gamma \rightarrow \Box_a\psi$ if and only if for every $\gamma' \in \Gamma_a$: $\vdash_{\mathbf{S5}} \gamma' \rightarrow \psi$.

Proposition 2.51 *Every formula of $\mathcal{L}$ is equivalent to a disjunction of explicit formulae under the semantics of S5.*

This is shown by Hales, French and Davies [21].

Definition 2.52 [Axiomatisation $\mathbf{AAML_{S5}}$] The axiomatisation $\mathbf{AML_{S5}}$ is a substitution schema consisting of the rules and axioms of $\mathbf{AML_{S5}}$ along with the rules and axioms **R**, **RP** and **NecR** of $\mathbf{AAML_K}$ and the axioms:

$$\begin{array}{rl} \mathbf{RS5} & \exists(\gamma^0 \wedge \nabla_a \Gamma_a) \leftrightarrow \exists\gamma^0 \wedge \bigwedge_{\gamma \in \Gamma_a} \Diamond_a \exists\gamma \\ \mathbf{RDist} & \exists(\gamma^0 \wedge \bigwedge_{a \in A} \nabla_a \Gamma_a) \leftrightarrow \bigwedge_{a \in A} \exists(\gamma^0 \wedge \nabla_a \Gamma_a) \end{array}$$

where $\gamma^0 \wedge \bigwedge_{a \in A} \exists \nabla_a \Gamma_a$ is an explicit formula and for every $a \in A$, $\gamma^0 \wedge \nabla_a \Gamma_a$ is an explicit formula.

Proposition 2.53 *The axiomatisation* $\mathbf{AAML_{S5}}$ *is sound and complete for the logic* $S5_{\otimes\forall}$.

Proposition 2.54 *The logic* $S5_{\otimes\forall}$ *is expressively equivalent to the logic S5.*

These results follow from similar reasoning to the same results in $K_{\otimes\forall}$.

3 Syntax

Definition 3.1 [Language of arbitrary action formula logic] The language $\mathcal{L}_{?\forall}$ of arbitrary action formula logic is inductively defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid [\alpha]\varphi \mid \forall\varphi$$

where $p \in P$, $a \in A$ and $\alpha \in \mathcal{L}_{?\forall}{}^{\mathrm{act}}$, and where the language $\mathcal{L}_{?\forall}{}^{\mathrm{act}}$ of arbitrary action formulae is inductively as:

$$\alpha ::= ?\varphi \mid \alpha \sqcup \alpha \mid \alpha \otimes \alpha \mid L_B(\alpha, \alpha)$$

where $\varphi \in \mathcal{L}_{?\forall}$ and $\emptyset \subset B \subseteq A$.

We use all of the standard abbreviations for arbitrary action model logic, in addition to the abbreviations $L_B\alpha ::= L_B(\alpha, \alpha)$ and $L_a(\alpha, \beta) ::= L_{\{a\}}(\alpha, \beta)$.

We denote non-deterministic choice ($\sqcup$) over a finite set of action formula $\Delta \subseteq \mathcal{L}_{?\forall}{}^{\mathrm{act}}$ by $\bigsqcup \Delta$ and we denote sequential execution ($\otimes$) of a finite, non-empty sequence of action formulae $(\alpha_i)_{i=0}^n \in \mathbb{N}^{\mathcal{L}_{?\forall}{}^{\mathrm{act}}}$ by $\bigotimes(\alpha_i)_{i=0}^n$ and define them in the obvious way.

We refer to the languages $\mathcal{L}_?$ of action formula logic and $\mathcal{L}_?{}^{\mathrm{act}}$ of action formulae, which are $\mathcal{L}_{?\forall}$ and $\mathcal{L}_{?\forall}{}^{\mathrm{act}}$ respectively, both without the $\forall$ operator,

As in the action model logic [7], the intended meaning of the operator $[\alpha]\varphi$ is that "φ is true in the result of any successful execution of the action α". In the following section we define the semantics of the action formula logic in terms of action model execution. For each setting of $\mathcal{K}$, $\mathcal{K}45$ and $\mathcal{S}5$ we provide a function $\tau_{\mathcal{C}} : \mathcal{L}_?{}^{\mathrm{act}} \to \mathcal{AM}$ of translating action formulae from $\mathcal{L}_?{}^{\mathrm{act}}$ into action models. The result of executing an action $\alpha \in \mathcal{L}_?{}^{\mathrm{act}}$ is determined by translating α into an action model $\tau_{\mathcal{C}}(\alpha) \in \mathcal{AM}_{\mathcal{C}}$, and then executing the action model in the usual way.

In each setting we have attempted to define the translation from action formulae into action models in such a way that the action formulae carry an intuitive description of the action that is performed by the corresponding action model. We call the ? operator the test operator, and describe the action $?\varphi$ as a test for φ. A test is intended to restrict the states in which an action can successfully execute to states where the condition φ is true initially, but otherwise leaves the state unchanged. We call the $\sqcup$ operator the non-deterministic choice operator, and describe the action $\alpha \sqcup \beta$ as a non-deterministic choice between α and β. We call the $\otimes$ operator the sequential execution operator, and describe the action $\alpha \otimes \beta$ as an execution of α followed by β. Finally we call L_B the learning operator, and describe the action $L_B(\alpha, \beta)$ as the agents in B learning that the actions α or β occurred. This action is intended to result in the agents B knowing or believing what would be true if α or β were executed. For example, if a consequence of executing α is that φ is true in the result, then the intention is that a consequence of executing $L_a(\alpha, \alpha)$ is that $\Box_a \varphi$ is true in the result. As we will see, this property is generally true in $K_?$, however due to the extra frame conditions of $\mathcal{K45}$ and $\mathcal{S5}$ it is only true for some formulae φ in $K45_?$ and $S5_?$.

Example 3.2 If p stands for the proposition "the grant application was successful" then the action described in Example 1.1 might be written in the form of an action formula as:

$$\begin{aligned}\alpha = & L_{Ed}(?p) \otimes \\ & L_{James}(L_{Ed}?p \sqcup L_{Ed}?\neg p \sqcup L_{Tim}?p \sqcup L_{Tim}?\neg p) \otimes \\ & L_{Tim}((?\neg p \otimes L_{James}?\neg p) \sqcup ?\top)\end{aligned}$$

4 Semantics

We now define the semantics of arbitrary action formula logic. As mentioned earlier, the semantics are defined by translating action formulae into action models. The translation used varies in each class of $\mathcal{K}$, $\mathcal{K45}$ and $\mathcal{S5}$ that we work in, according to the frame conditions in each class. Therefore our semantics are parameterised by a function $\tau_{\mathcal{C}} : \mathcal{L}_?^{\text{act}} \to \mathcal{AM}$ that will vary according to the class of Kripke models.

Definition 4.1 [Semantics of arbitrary action formula logic] Let $\mathcal{C}$ be a class of Kripke models, let $\tau_{\mathcal{C}} : \mathcal{L}_?^{\text{act}} \to \mathcal{AM}$ be a function from action formulae to multi-pointed action models, and let $M = (S, R, V) \in \mathcal{C}$ be a Kripke model.

Then the interpretation of $\varphi \in \mathcal{L}_{?\forall}$ in the logic $C_{?\forall}$ is the same as its interpretation in modal logic given in Definition 2.5, with the additional inductive cases:

$$M_s \models [\alpha]\varphi \text{ iff } M_s \otimes \tau_{\mathcal{C}}(\alpha) \in \mathcal{C} \text{ implies } M_s \otimes \tau_{\mathcal{C}}(\alpha) \models \varphi$$
$$M_s \models \forall\varphi \text{ iff for every } M'_{s'} \in \mathcal{C} \text{ such that } M'_{s'} \preceq M_s : M'_{s'} \models \varphi$$

where action model execution $\otimes$ is as defined in Definition 2.18 and the refinement relation is defined in Definition 2.36.

We note that the semantics of arbitrary action formula logic $C_{?\forall}$ are very similar to the semantics of arbitrary action model logic $C_{\otimes\forall}$ [19]. We generalise the semantics to the classes of $\mathcal{K}$, $\mathcal{K}45$ and $\mathcal{S}5$ by introducing the parameterised class $\mathcal{C}$ and restricting successful updates to those that result in $\mathcal{C}$ models as in the approach of Balbiani, et al [5]. The difference is that as actions are specified in $\mathcal{L}_{?\forall}$ formulae as action formulae, then the semantics must first translate the action formulae into action models before performing action model execution. As such there is a semantically correct translation from $\mathcal{L}_{?\forall}$ formulae to $\mathcal{L}_{\otimes\forall}$ formulae (by replacing occurrences of α with $\tau_{\mathcal{C}}(\alpha)$), and any validities, axioms or results from arbitrary action model logic also apply in this setting if the language is restricted to action models that are defineable by action formulae. Therefore for the current section and the following sections concering the axiomatisations (Section 5) and correspondence results (Section 6), we will deal only with the action formula logic, rather than the full arbitrary action formula logic, focussing on the differences and correspondences between action formulae and action models, rather than getting distracted by the refinement quantifiers which behave identically between each logic. We return to the full arbitrary action formula logic in Section 7 for the synthesis results.

We give the following general result.

Proposition 4.2 *Let $\mathcal{C}$ be a class of Kripke models. For every $\varphi \in \mathcal{L}_{?\forall}$ there exists $\varphi' \in \mathcal{L}_{\otimes\forall}$ such that for every $M_T \in \mathcal{C}$: $M_T \vDash_{C_{?\forall}} \varphi$ if and only if $M_T \vDash_{C_{\otimes\forall}} \varphi'$.*

In the following subsections we will give definitions for $\tau_{\mathcal{K}}$, $\tau_{\mathcal{K}45}$ and $\tau_{\mathcal{S}5}$. These functions vary according to the class of Kripke models being used. When the class is clear from context, then we will simply write τ instead of $\tau_{\mathcal{C}}$.

We begin by giving a definition of τ for translating actions involving non-deterministic choice and sequential execution. These definitions are common to all of the settings we are working in.

Definition 4.3 [Non-deterministic choice] Let $\mathcal{C} \in \{\mathcal{K}, \mathcal{K}45, \mathcal{S}5\}$ and let $\alpha, \beta \in \mathcal{L}_?^{\text{act}}$ where $\tau_{\mathcal{C}}(\alpha) = \mathsf{M}^{\alpha}_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$ and $\tau_{\mathcal{C}}(\beta) = \mathsf{M}^{\beta}_{\mathsf{T}^\beta} = ((\mathsf{S}^\beta, \mathsf{R}^\beta, \mathsf{pre}^\beta), \mathsf{T}^\beta)$ such that S^α and S^β are disjoint. We define $\tau_{\mathcal{C}}(\alpha \sqcup \beta) = \mathsf{M}_{\mathsf{T}} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$\begin{aligned} \mathsf{S} &= \mathsf{S}^\alpha \cup \mathsf{S}^\beta \\ \mathsf{R}_a &= \mathsf{R}^\alpha_a \cup \mathsf{R}^\beta_a \text{ for } a \in A \\ \mathsf{pre} &= \mathsf{pre}^\alpha \cup \mathsf{pre}^\beta \\ \mathsf{T} &= \mathsf{T}^\alpha \cup \mathsf{T}^\beta \end{aligned}$$

Definition 4.4 [Sequential execution] Let $\mathcal{C} \in \{\mathcal{K}, \mathcal{K}45, \mathcal{S}5\}$, and let $\alpha, \beta \in \mathcal{L}_?^{\text{act}}$ where $\tau_{\mathcal{C}}(\alpha) = \mathsf{M}^{\alpha}_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$ and $\tau_{\mathcal{C}}(\beta) = \mathsf{M}^{\beta}_{\mathsf{T}^\beta} = ((\mathsf{S}^\beta, \mathsf{R}^\beta, \mathsf{pre}^\beta), \mathsf{T}^\beta)$. We define $\tau_{\mathcal{C}}(\alpha \otimes \beta) = \mathsf{M}^{\alpha}_{\mathsf{T}^\alpha} \otimes \mathsf{M}^{\beta}_{\mathsf{T}^\beta}$.

We give some properties of non-deterministic choice and sequential execution of action formulae.

Proposition 4.5 *Let $\alpha, \beta, \gamma \in \mathcal{L}_?{}^{act}$ and $\varphi \in \mathcal{L}_?$. Then the following are valid in $K_?$, $K45_?$ and $S5_?$:*

$$\models [\alpha \sqcup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi)$$
$$\models [\alpha \otimes \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$$
$$\models [\alpha \sqcup \alpha]\varphi \leftrightarrow [\alpha]\varphi$$
$$\models [\alpha \sqcup \beta]\varphi \leftrightarrow [\beta \sqcup \alpha]\varphi$$
$$\models [(\alpha \sqcup \beta) \sqcup \gamma]\varphi \leftrightarrow [\alpha \sqcup (\beta \sqcup \gamma)]\varphi$$
$$\models [(\alpha \otimes \beta) \otimes \gamma]\varphi \leftrightarrow [\alpha \otimes (\beta \otimes \gamma)]\varphi$$
$$\models [(\alpha \sqcup \beta) \otimes \gamma]\varphi \leftrightarrow [(\alpha \otimes \gamma) \sqcup (\beta \otimes \gamma)]\varphi$$

These validities follow trivially from the semantics of $C_{?\forall}$ and Definitions 4.3 and 4.4.

In the following subsections we give definitions of τ_C for translating action formulae involving tests and learning in the settings of $\mathcal{K}$, $\mathcal{K}45$ and $\mathcal{S}5$. We note that in each subsection the constructions of action models used to define tests and learning closely resemble the constructions of refinements used to show the soundness of axioms in refinement modal logic [10,21].

4.1 $\mathcal{K}$

Definition 4.6 [Test] Let $\varphi \in \mathcal{L}_?$. We define $\tau(?\varphi) = \mathsf{M}_\mathsf{T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$\mathsf{S} = \{\mathsf{test}, \mathsf{skip}\}$$
$$\mathsf{R}_a = \{(\mathsf{test}, \mathsf{skip}), (\mathsf{skip}, \mathsf{skip})\} \text{ for } a \in A$$
$$\mathsf{pre} = \{(\mathsf{test}, \varphi), (\mathsf{skip}, \top)\}$$
$$\mathsf{T} = \{\mathsf{test}\}$$

Definition 4.7 [Learning] Let $\alpha \in \mathcal{L}_?{}^{\mathrm{act}}$ where $\tau(\alpha) = \mathsf{M}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$. Let test and skip be new states not appearing in S^α. We define $\tau(L_B(\alpha, \alpha)) = \mathsf{M}_\mathsf{T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$\mathsf{S} = \mathsf{S}^\alpha \cup \{\mathsf{test}, \mathsf{skip}\}$$
$$\mathsf{R}_a = \mathsf{R}^\alpha_a \cup \{(\mathsf{skip}, \mathsf{skip})\} \cup \{(\mathsf{test}, \mathsf{t}^\alpha) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \text{ for } a \in B$$
$$\mathsf{R}_a = \mathsf{R}^\alpha_a \cup \{(\mathsf{test}, \mathsf{skip}), (\mathsf{skip}, \mathsf{skip})\} \text{ for } a \notin B$$
$$\mathsf{pre} = \mathsf{pre}^\alpha \cup \{(\mathsf{test}, \top), (\mathsf{skip}, \top)\}$$
$$\mathsf{T} = \{\mathsf{test}\}$$

We define $\tau(L_B(\alpha, \beta)) = \tau(L_B(\alpha \sqcup \beta, \alpha \sqcup \beta))$.

We note that the syntax of action formula logic defines the learning operator as a binary operator that can be applied to two different action formulae, however in the setting of $\mathcal{K}$ and $\mathcal{K}45$ we only give a direct definition of τ for actions of the form $L_B(\alpha, \alpha)$ and define the more general case in terms of this. Intuitively $L_B(\alpha, \beta)$ is intended to represent an action where the agents in B

learn that α or β have occurred (i.e. that $\alpha \sqcup \beta$ has occurred). The setting of $\mathcal{S5}$ corresponds to a notion of *knowledge*, where anything that an agent *knows* must be true, and therefore anything that an agent *learns* must also be true. So in an action where agents learn that α or β have occurred, one of those actions must have actually occurred. Therefore in $\mathcal{S5}$ we describe the action $L_B(\alpha, \beta)$ as the agents in B learning that α or β have occurred, when in reality α has actually occurred. On the other hand, the settings of $\mathcal{K}$ and $\mathcal{K45}$ correspond more closely to a notion of *belief*, where there is no requirement that what an agent *believes* is true. So in an action where agents learn that α or β have occurred, neither of these actions must actually have occurred. Therefore in the settings of $\mathcal{K}$ and $\mathcal{K45}$ we make no distinction between α and β in a description of the action $L_B(\alpha, \beta)$, hence the definition of τ given in these settings.

4.2 $\mathcal{K45}$

Definition 4.8 [Test] Let $\varphi \in \mathcal{L}_?$. We define $\tau(?\varphi)$ as in Definition 4.6 for $\mathcal{K}$.

Definition 4.9 [Learning] Let $\alpha \in \mathcal{L}_?{}^{\text{act}}$ where $\tau(\alpha) = \mathsf{M}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$. Let test and skip be new states not appearing in S^α. For every $\mathsf{t}^\alpha \in \mathsf{T}^\alpha$ let $\bar{\mathsf{t}}^\alpha$ be a new state not appearing in S^α. We call each $\bar{\mathsf{t}}^\alpha$ a *proxy state* for t^α. We define $\tau(L_B(\alpha, \alpha)) = \mathsf{M}_\mathsf{T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$
\begin{aligned}
\mathsf{S} &= \mathsf{S}^\alpha \cup \{\mathsf{test}, \mathsf{skip}\} \cup \{\bar{\mathsf{t}}^\alpha \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \\
\mathsf{R}_a &= \mathsf{R}^\alpha_a \cup \{(\mathsf{skip}, \mathsf{skip})\} \cup \{(\mathsf{test}, \bar{\mathsf{t}}^\alpha) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \cup \\
&\qquad \{(\bar{\mathsf{t}}^\alpha, \bar{\mathsf{u}}^\alpha) \mid \mathsf{t}^\alpha, \mathsf{u}^\alpha \in \mathsf{T}^\alpha\} \text{ for } a \in B \\
\mathsf{R}_a &= \mathsf{R}^\alpha_a \cup \{(\mathsf{test}, \mathsf{skip}), (\mathsf{skip}, \mathsf{skip})\} \cup \\
&\qquad \{(\bar{\mathsf{t}}^\alpha, \mathsf{u}^\alpha) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha, \mathsf{u}^\alpha \in \mathsf{t}^\alpha \mathsf{R}^\alpha_a\} \text{ for } a \notin B \\
\mathsf{pre} &= \mathsf{pre}^\alpha \cup \{(\mathsf{test}, \top), (\mathsf{skip}, \top)\} \cup \{(\bar{\mathsf{t}}^\alpha, \mathsf{pre}^\alpha(\mathsf{t}^\alpha)) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \\
\mathsf{T} &= \{\mathsf{test}\}
\end{aligned}
$$

As in Definition 4.7, we define $\tau(L_B(\alpha, \beta)) = \tau(L_B(\alpha \sqcup \beta, \alpha \sqcup \beta))$.

Lemma 4.10 *Let $\alpha \in \mathcal{L}_?{}^{act}$. Then $\tau(\alpha) \in \mathcal{AM}_{\mathcal{K45}}$.*

Lemma 4.11 *Let $\alpha \in \mathcal{L}_?{}^{act}$ and let $M_T \in \mathcal{K45}$. Then $M_T \otimes \tau(\alpha) \in \mathcal{K45}$.*

We note that the definition for τ given here varies considerably from the definition given in the setting of $\mathcal{K}$ due to the presence of the proxy states. The proxy states are introduced due to the additional frame constraints in $\mathcal{K45}$ and the desire that the action models constructed by τ be $\mathcal{AM}_{\mathcal{K45}}$ action models. In constructing $\tau(L_B\alpha)$ we wish to construct an action model with a root state whose B-successors are the root states of $\tau(\alpha)$, so that the result of executing the action $L_B\alpha$ is that the agents B believe that the action α has occurred. However in order for this construction to result in a $\mathcal{AM}_{\mathcal{K45}}$ action model, we must take the transitive, Euclidean closure of the B-successors of the root state. If we were to perform a construction similar to that used in the setting of $\mathcal{K}$ where proxy states are not used, then this would mean that the for every $b \in B$, the b-successors of the root state would include all of the

b-successors of the root states, and not just the root states themselves. To show why this is not desireable, consider the simple example of the action $L_a?\varphi$. The intention is that this action represents a private announcement to a that φ is true, as it is in the setting of $\mathcal{K}$. Without using proxy states, if we wanted to include the state test in the a-successors of the root state of $\tau(\alpha)$ then in order to construct a $\mathcal{AM}_{\mathcal{K}45}$ action model we would need to take the transitive, Euclidean closure of the a-successors of test. As skip is an a-successor of test in the action $?\varphi$, then this would mean that a would not be able to distinguish between the actions states test and skip and so the result of executing $\tau(\alpha)$ would be that a learns nothing. With the construction provided, the action $L_a?\varphi$ gives the desired result that a learns that φ is true.

We also note that the results presented in this paper for $\mathcal{K}45$ can be extended to $\mathcal{KD}45$ by modifying Definition 4.9 so that $\mathsf{pre}(\mathsf{test}) = \bigwedge_{a\in B}\bigvee_{\mathsf{t}^\alpha\in\mathsf{T}^\alpha}\Diamond_a\mathsf{pre}^\alpha(\mathsf{t}^\alpha)$, which guarantees that the result of successfully executing an action formula has the seriality property of $\mathcal{KD}45$.

4.3 $\mathcal{S}5$

Definition 4.12 [Test] Let $\varphi \in \mathcal{L}_?$. We define $\tau(?\varphi) = \mathsf{M}_\mathsf{T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$
\begin{aligned}
\mathsf{S} &= \{\mathsf{test}, \mathsf{skip}\}\\
\mathsf{R}_a &= \mathsf{S}^2 \text{ for } a \in A\\
\mathsf{pre} &= \{(\mathsf{test}, \varphi), (\mathsf{skip}, \top)\}\\
\mathsf{T} &= \{\mathsf{test}\}
\end{aligned}
$$

Definition 4.13 [Learning] Let $\alpha, \beta \in \mathcal{L}_?{}^{\text{act}}$ where $\tau(\alpha) = \mathsf{M}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$ and $\tau(\beta) = \mathsf{M}^\beta_{\mathsf{T}^\beta} = ((\mathsf{S}^\beta, \mathsf{R}^\beta, \mathsf{pre}^\beta), \mathsf{T}^\beta)$. For every $\mathsf{t} \in \mathsf{T}^\alpha \cup \mathsf{T}^\beta$ let $\bar{\mathsf{t}}$ be a new state not appearing in $\mathsf{S}^\alpha \cup \mathsf{S}^\beta$. We define $\tau(L_B(\alpha, \beta)) = \mathsf{M}_\mathsf{T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$
\begin{aligned}
\mathsf{S} &= \mathsf{S}^\alpha \cup \mathsf{S}^\beta \cup \{\bar{\mathsf{t}} \mid \mathsf{t} \in \mathsf{T}^\alpha \cup \mathsf{T}^\beta\}\\
\mathsf{R}_a &= \mathsf{R}^\alpha_a \cup \mathsf{R}^\beta_a \cup \{(\bar{\mathsf{t}}, \bar{\mathsf{u}}) \mid \mathsf{t}, \mathsf{u} \in \mathsf{T}^\alpha \cup \mathsf{T}^\beta\} \text{ for } a \in B\\
\mathsf{R}_a &= \mathsf{R}^\alpha_a \cup \mathsf{R}^\beta_a \cup \bigcup_{\mathsf{t}\in\mathsf{T}^\alpha\cup\mathsf{T}^\beta} (\{\bar{\mathsf{t}}\} \cup \mathsf{t}(\mathsf{R}^\alpha_a \cup \mathsf{R}^\beta_a))^2 \text{ for } a \notin B\\
\mathsf{pre} &= \mathsf{pre}^\alpha \cup \mathsf{pre}^\beta \cup \{(\bar{\mathsf{t}}, (\mathsf{pre}^\alpha \cup \mathsf{pre}^\beta)(\mathsf{t})) \mid \mathsf{t} \in \mathsf{T}^\alpha \cup \mathsf{T}^\beta\}\\
\mathsf{T} &= \{\bar{\mathsf{t}} \mid \mathsf{t} \in \mathsf{T}^\alpha\}
\end{aligned}
$$

Lemma 4.14 *Let $\alpha \in \mathcal{L}_?{}^{act}$. Then $\tau(\alpha) \in \mathcal{AM}_{\mathcal{S}5}$.*

Lemma 4.15 *Let $\alpha \in \mathcal{L}_?{}^{act}$ and let $M_T \in \mathcal{S}5$. Then $M_T \otimes \tau(\alpha) \in \mathcal{S}5$.*

We note that as in the setting of $\mathcal{K}45$ the definition of τ uses proxy states to construct action models from learning operators. However unlike in the settings of $\mathcal{K}$ and $\mathcal{K}45$ this construction does not introduce the new states test and skip. As discussed earlier this is because in the setting of $\mathcal{S}5$, in an action where agents learn that α or β have occurred, one of those actions must have actually occurred. Unlike in the settings of $\mathcal{K}$ and $\mathcal{K}45$ we have distinguished between the actions α and β, designating that α is the action that has actually occurred. We also note that the definition of τ for test operators is different from that used in $\mathcal{K}$ and $\mathcal{K}45$, simply to account for the additional frame constraints of $\mathcal{S}5$.

5 Axiomatisation

In the following subsections we give sound and complete axiomatisations for the action formulae logic in the settings of $\mathcal{K}$ and $\mathcal{K}45$. In the setting of $\mathcal{S}5$ we provide a sound but not complete axiomatisation, and comment on the difficulty of giving a complete axiomatisation and the possible alternatives. We note that axiomatisations for arbitrary action formula logic in these settings can be derived trivially from these axiomatisations by adding the additional axioms and rules from refinement modal logic.

5.1 $\mathcal{K}$

Definition 5.1 [Axiomatisation $\mathbf{AFL_K}$] The axiomatisation $\mathbf{AFL_K}$ is a substitution schema consisting of the rules and axioms of **K** along with the axioms:

$$
\begin{aligned}
\mathbf{LT} &\vdash [?\varphi]\psi \leftrightarrow (\varphi \to \psi) \text{ for } \psi \in \mathcal{L} \\
\mathbf{LU} &\vdash [\alpha \sqcup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi) \\
\mathbf{LS} &\vdash [\alpha \otimes \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi \\
\mathbf{LP} &\vdash [L_B(\alpha,\beta)]p \leftrightarrow p \\
\mathbf{LN} &\vdash [L_B(\alpha,\beta)]\neg\varphi \leftrightarrow \neg[L_B(\alpha,\beta)]\varphi \\
\mathbf{LC} &\vdash [L_B(\alpha,\beta)](\varphi \wedge \psi) \leftrightarrow ([L_B(\alpha,\beta)]\varphi \wedge [L_B(\alpha,\beta)]\psi) \\
\mathbf{LK1} &\vdash [L_B(\alpha,\beta)]\Box_a\varphi \leftrightarrow \Box_a[\alpha \sqcup \beta]\varphi \text{ for } a \in B \\
\mathbf{LK2} &\vdash [L_B(\alpha,\beta)]\Box_a\varphi \leftrightarrow \Box_a\varphi \text{ for } a \notin B
\end{aligned}
$$

and the rule:

$$\mathbf{NecL} \text{ From } \vdash \varphi \text{ infer } \vdash [\alpha]\varphi$$

Proposition 5.2 *The axiomatisation* $\mathbf{AFL_K}$ *is sound in the logic* $K_\otimes$*.*

Proof. **LT** follows from applying the reduction axioms of $\mathbf{AML_K}$ inductively to $[?\varphi]\psi$.

LU and **LS** follow from Proposition 4.5.

Let $\tau(L_b(\alpha,\beta)) = \mathsf{M_s} = ((\mathsf{S},\mathsf{R},\mathsf{pre}),\mathsf{s})$. **LP**, **LN** and **LC** follow trivially from the $\mathbf{AML_K}$ axioms **AP**, **AN** and **AC** respectively, noting from Definition 4.7 that $\mathsf{pre}(\mathsf{s}) = \top$. **LK1** follows trivially from the $\mathbf{AML_K}$ axiom **AK**, noting from Definition 4.7 that as $a \in A$ then $\mathsf{M}_{\mathsf{sR}_a} \leftrightarrows \tau(\alpha \sqcup \beta)$. **NecL** follows trivially from the $\mathbf{AML_K}$ rule **NecA**. **LK2** follows trivially from the $\mathbf{AML_K}$ axiom **AK**, noting from Definition 4.7 that as $a \notin A$ then $\mathsf{M}_{\mathsf{sR}_a} \leftrightarrows \tau(?\top)$. $\Box$

Proposition 5.3 *The axiomatisation* $\mathbf{AFL_K}$ *is complete for the logic* $K_\otimes$*.*

We note that the axiomatisation $\mathbf{AFL_K}$ forms a set of reduction axioms that gives a provably correct translation from $\mathcal{L}_?$ to $\mathcal{L}$.

Example 5.4 We give an example derivation that the action formula α given in Example 3.2 does indeed satisfy (part of) the epistemic goal stated in Example 1.1.

$$\vdash [?p]p \leftrightarrow (p \to p) \tag{1}$$
$$\vdash [?p]p \tag{2}$$
$$\vdash \Box_{Ed}[?p]p \tag{3}$$
$$\vdash [L_{Ed}?p]\Box_{Ed}p \tag{4}$$

(1) follows from **LT**, (3) follows from **NecK** and (4) follows from **LK1**.

Similarly we have

$$\vdash [L_{Ed}?\neg p]\Box_{Ed}\neg p$$
$$\vdash [L_{Tim}?p]\Box_{Tim}p$$
$$\vdash [L_{Tim}?\neg p]\Box_{Tim}\neg p$$

Let $\varphi = \Box_{Ed}p \vee \Box_{Ed}\neg p \vee \Box_{Tim}p \vee \Box_{Tim}\neg p$. Then:

$$\vdash [L_{Ed}?p \sqcup L_{Ed}?\neg p \sqcup L_{Tim}?p \sqcup L_{Tim}?\neg p]\varphi \tag{5}$$
$$\vdash \Box_{James}[L_{Ed}?p \sqcup L_{Ed}?\neg p \sqcup L_{Tim}?p \sqcup L_{Tim}?\neg p]\varphi \tag{6}$$
$$\vdash [L_{James}(L_{Ed}?p \sqcup L_{Ed}?\neg p \sqcup L_{Tim}?p \sqcup L_{Tim}?\neg p)]\Box_{James}\varphi \tag{7}$$
$$\vdash [\alpha]\Box_{James}\varphi \tag{8}$$

(5) follows from **LU**, (6) follows from **NecK** and (7) follows from **LK1**. (8) follows from **LS** and **LK2**.

Therefore a consequence of successfully executing α is that James learns that Ed or Tim knows whether the grant application was successful.

5.2 $\mathcal{K}45$

Definition 5.5 [Axiomatisation $\mathbf{AFL_{K45}}$] The axiomatisation $\mathbf{AFL_{K45}}$ is a substitution schema consisting of the rules and axioms of **K45** along with the rules and axioms of $\mathbf{AFL_{K}}$, but substituting the $\mathbf{AFL_{K}}$ axiom **LK1** for the axiom:

$$\textbf{LK1} \vdash [L_B(\alpha, \beta)]\Box_a\chi \leftrightarrow \Box_a[\alpha \sqcup \beta]\chi \text{ for } a \in B$$

and the rule:

$$\textbf{NecL} \text{ From } \vdash \varphi \text{ infer } \vdash [\alpha]\varphi$$

where χ is a $(A \setminus \{a\})$-restricted formula.

Proposition 5.6 *The axiomatisation* $\mathbf{AFL_{K45}}$ *is sound in the logic* $K45_{\otimes}$.

Proof. Soundness of **LT**, **LU**, **LS**, **LP**, **LN**, **LC**, **LK2** and **NecL** follow from the same reasoning as in the proof of Proposition 5.2.

LK1 follows from the $\mathbf{AML_{K45}}$ axiom **AK**. We note that as $a \in B$, from Definition 4.9 we have $\mathsf{M}_{\mathsf{sR}_a} \leftrightarrows_{(A\setminus\{a\})} \tau(\alpha \sqcup \beta)$, and as χ is $(A \setminus \{a\})$-restricted formula then $\models [\mathsf{M}_{\mathsf{sR}_a}]\chi \leftrightarrow [\tau(\alpha \sqcup \beta)]\chi$. □

Proposition 5.7 *The axiomatisation* $\mathbf{AFL_{K45}}$ *is complete for the logic* $K45_{\otimes}$.

We note that the axiomatisation $\mathbf{AFL_{K45}}$ forms a set of reduction axioms that gives a provably correct translation from $\mathcal{L}_?$ to $\mathcal{L}$. To translate a subformula $[\alpha]\varphi$, where $\varphi \in \mathcal{L}$, we must first translate φ to the alternating disjunctive normal form of [21], which gives the property that for every subformula $\Box_a\psi$, the formula ψ is $(A \setminus \{a\})$-restricted, and therefore **LK1** is applicable.

5.3 $\mathcal{S5}$

Definition 5.8 [Axiomatisation $\mathbf{AFL_{S5}}$] The axiomatisation $\mathbf{AFL_{S5}}$ is a substitution schema consisting of the rules and axioms of **S5** along with the axioms:

$$\begin{array}{rl}
\mathbf{LT} & \vdash [?\varphi]\psi \leftrightarrow (\varphi \rightarrow \psi) \text{ for } \psi \in \mathcal{L} \\
\mathbf{LU} & \vdash [\alpha \sqcup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi) \\
\mathbf{LS} & \vdash [\alpha \otimes \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi \\
\mathbf{LP} & \vdash [L_B(\alpha, \beta)]p \leftrightarrow p \\
\mathbf{LN} & \vdash [L_B(\alpha, \beta)]\neg\varphi \leftrightarrow \neg[L_B(\alpha, \beta)]\varphi \\
\mathbf{LC} & \vdash [L_B(\alpha, \beta)](\varphi \wedge \psi) \leftrightarrow ([L_B(\alpha, \beta)]\varphi \wedge [L_B(\alpha, \beta)]\psi)
\end{array}$$

and the rule:

$$\mathbf{NecL} \text{ From } \vdash \varphi \text{ infer } \vdash [\alpha]\varphi$$

where χ is a $(A \setminus \{a\})$-restricted formula.

Proposition 5.9 *The axiomatisation* $\mathbf{AFL_{S5}}$ *is sound in the logic* $\mathcal{S5}_\otimes$.

Proof. Soundness of **LT**, **LU**, **LS**, **LP**, **LN**, **LC** and **NecL** follow from the same reasoning as in the proof of Proposition 5.2. □

We note that we do not have a axioms in $\mathcal{S5}_?$ corresponding to the axioms **LK1** and **LK2** from $\mathbf{AFL_K}$ and $\mathbf{AFL_{K45}}$. **LK1** works in the setting of $\mathcal{K}$ because the B-successors of the root state of $\tau(L_B\alpha)$ are bisimilar to the root states of $\tau(\alpha)$, and so the consequences of executing $\tau(\alpha)$ are the same as the consequences of executing the B-successors of $\tau(L_B\alpha)$. In the setting of $\mathcal{K45}$ this is not the case, however we do have the restricted property of B-bisimilarity, giving us that the B-restricted consequences are the same. In the setting of $\mathcal{S5}$ we do not know of such a property to relate the consequences of the B-successors of $\tau(L_B(\alpha, \beta))$ to the consequences of $\tau(\alpha \sqcup \beta)$. Given the correspondence results of the previous section, it should be possible to construct an action formula that is n-bisimilar to the B-successors of $\tau(L_B(\alpha, \beta))$, where $d(\varphi) = n$, and define axioms for **LK1** and **LK2** in terms of this action formula and not $\alpha \sqcup \beta$. However translating $\mathcal{L}_?$ formulae into $\mathcal{L}_\otimes$ formulae and then using the axiomatisation $\mathbf{AML_{S5}}$ would certainly be simpler.

6 Correspondence

In the following subsections we show the correspondence between action formulae and action models in the settings of $\mathcal{K}$, $\mathcal{K45}$ and $\mathcal{S5}$. In each setting we show that action formulae are capable of representing any action model up to n-bisimilarity.

6.1 $\mathcal{K}$

To begin we give two lemmas to simplify the construction that we will use for our correspondence result in $\mathcal{K}$.

Lemma 6.1 *Let* $\varphi \in \mathcal{L}_?$ *and* $\mathsf{M}_\mathsf{s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{AM}$. *Then let* $\mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}') \in \mathcal{AM}$ *where:*

$$\begin{aligned} \mathsf{S}' &= \mathsf{S} \cup \{\mathsf{s}'\} \\ \mathsf{R}'_a &= \mathsf{R}_a \cup \{(\mathsf{s}', \mathsf{t}) \mid \mathsf{t} \in \mathsf{s}\mathsf{R}_a\} \text{ for } a \in A \\ \mathsf{pre}' &= \mathsf{pre} \cup \{(\mathsf{s}', \varphi \wedge \mathsf{pre}(\mathsf{s}))\} \end{aligned}$$

Then $\tau(?\varphi) \otimes \mathsf{M}_\mathsf{s} \leftrightarroweq \mathsf{M}'_{\mathsf{s}'}$.

Lemma 6.2 *Let* $\alpha \in \mathcal{L}_?{}^{act}$ *where* $\tau(\alpha) = \mathsf{M}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$, $a \in A$ *and* $\mathsf{M}_\mathsf{s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{AM}$ *such that* $\mathsf{s}\mathsf{R}_a = \{\mathsf{t}\}$ *for some* $\mathsf{t} \in \mathsf{S}$ *and* $\mathsf{t}\mathsf{R}_a = \{\mathsf{t}\}$ *Then let* $\mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}') \in \mathcal{AM}$ *where:*

$$\begin{aligned} \mathsf{S}' &= \mathsf{S} \cup \mathsf{S}^\alpha \cup \{\mathsf{s}'\} \\ \mathsf{R}'_a &= \mathsf{R}_a \cup \mathsf{R}^\alpha_a \cup \{(\mathsf{s}', \mathsf{t}^\alpha) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \\ \mathsf{R}'_b &= \mathsf{R}_b \cup \mathsf{R}^\alpha_b \cup \{(\mathsf{s}', \mathsf{t}) \mid \mathsf{t} \in \mathsf{s}\mathsf{R}_b\} \text{ for } b \in A \setminus \{a\} \\ \mathsf{pre}' &= \mathsf{pre} \cup \{(\mathsf{s}', \mathsf{pre}(\mathsf{s}))\} \end{aligned}$$

Then $\tau(L_a\alpha) \otimes \mathsf{M}_\mathsf{s} \leftrightarroweq \mathsf{M}'_{\mathsf{s}'}$.

Proposition 6.3 *Let* $\mathsf{M}_\mathsf{s} \in \mathcal{AM}$ *and let* $n \in \mathbb{N}$. *Then there exists* $\alpha \in \mathcal{L}_?{}^{act}$ *such that* $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \tau(\alpha)$.

Proof. By induction on n.

Suppose that $n = 0$. Let $\alpha = ?\mathsf{pre}(\mathsf{s})$ and $\tau(\alpha) = \mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}')$. From Definition 4.6 we have that $\mathsf{pre}(\mathsf{s}) = \mathsf{pre}'(\mathsf{s}')$, so $(\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'})$ satisfies **atoms** and therefore $\mathsf{M}_\mathsf{s} \leftrightarroweq_0 \mathsf{M}'_{\mathsf{s}'}$.

Suppose that $n > 0$. By the induction hypothesis, for every $a \in A$, $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ there exists $\alpha^{a,\mathsf{t}} \in \mathcal{L}_?{}^{\mathrm{act}}$ such that $\mathsf{M}_\mathsf{t} \leftrightarroweq_{(n-1)} \tau(\alpha^{a,\mathsf{t}})$, where $\tau(\alpha^{a,\mathsf{t}}) \leftrightarroweq \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}} = ((\mathsf{S}^{a,\mathsf{t}}, \mathsf{R}^{a,\mathsf{t}}, \mathsf{pre}^{a,\mathsf{t}}), \mathsf{s}^{a,\mathsf{t}})$.

Let $\alpha = ?\mathsf{pre}(\mathsf{s}) \otimes \bigotimes_{a \in A} L_a(\bigsqcup_{\mathsf{t} \in \mathsf{s}\mathsf{R}_a} \alpha^\mathsf{t})$. Then from Lemmas 6.1 and 6.2: $\tau(\alpha) \leftrightarroweq \mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}')$ where:

$$\begin{aligned} \mathsf{S}' &= \bigcup_{a \in A, \mathsf{t} \in \mathsf{s}\mathsf{R}_a} (\mathsf{S}^{a,\mathsf{t}}) \cup \{\mathsf{s}'\} \\ \mathsf{R}'_a &= \bigcup_{b \in A, \mathsf{t} \in \mathsf{s}\mathsf{R}_b} (\mathsf{R}^{b,\mathsf{t}}_a) \cup \{(\mathsf{s}', \mathsf{s}^{a,\mathsf{t}}) \mid \mathsf{t} \in \mathsf{s}\mathsf{R}_a\} \text{ for } a \in A \\ \mathsf{pre}' &= \bigcup_{a \in A, \mathsf{t} \in \mathsf{s}\mathsf{R}_a} (\mathsf{pre}^{a,\mathsf{t}}) \cup \{(\mathsf{s}', \mathsf{pre}(\mathsf{s}))\} \end{aligned}$$

We note for every $a \in A$, $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ that $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarroweq \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}}$ as for every $a \in A$, $\mathsf{u} \in \mathsf{S}^{a,\mathsf{t}}$ we have $\mathsf{u}\mathsf{R}'_a = \mathsf{u}\mathsf{R}^{a,\mathsf{t}}_a$.

We show that $(\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'})$ satisfies **atoms**, **forth-n-a** and **back-n-a** for every $a \in A$.

atoms By construction $\mathsf{pre}'(\mathsf{s}') = \mathsf{pre}(\mathsf{s})$.

forth-n-a Let $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$. By construction $\mathsf{s}^{a,\mathsf{t}} \in \mathsf{s}'\mathsf{R}'_a$, by the induction hypothesis $\mathsf{M}_\mathsf{t} \leftrightarrows_{(n-1)} \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}}$ and from above $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarrows \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$. Therefore by transitivity $\mathsf{M}_\mathsf{t} \leftrightarrows_{(n-1)} \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$.

back-n-a Follows from similar reasoning to **forth-n-a**.

Therefore $\mathsf{M}_\mathsf{s} \leftrightarrows_n \tau(\alpha)$. □

Corollary 6.4 *Let $\mathsf{M}_\mathsf{s} \in \mathcal{AM}$. Then for every $\varphi \in \mathcal{L}_\otimes$ there exists $\alpha \in \mathcal{L}_?{}^{act}$ such that $\models_{K_\otimes} [\mathsf{M}_\mathsf{s}]\varphi \leftrightarrow [\tau(\alpha)]\varphi$.*

Proof. Suppose that $d(\varphi) = n$. From Proposition 6.3 there exists $\alpha \in \mathcal{L}_?{}^{\text{act}}$ such that $\mathsf{M}_\mathsf{s} \leftrightarrows_n \tau(\alpha)$. Therefore for every $M_\mathsf{s} \in \mathcal{K}$ we have $M_\mathsf{s} \otimes \mathsf{M}_\mathsf{s} \leftrightarrows_n M_\mathsf{s} \otimes \tau(\alpha)$ and so $M_\mathsf{s} \otimes \mathsf{M}_\mathsf{s} \models_{K_\otimes} \varphi$ if and only if $M_\mathsf{s} \otimes \tau(\alpha) \models_{K_\otimes} \varphi$. Therefore $M_\mathsf{s} \models_{K_\otimes} [\mathsf{M}_\mathsf{s}]\varphi$ if and only if $M_\mathsf{s} \models_{K_\otimes} [\tau(\alpha)]\varphi$. □

Corollary 6.5 *Let $\varphi \in \mathcal{L}_\otimes$. Then there exists $\varphi' \in \mathcal{L}_?$ such that for every $M_s \in \mathcal{K}$: $M_s \models_{K_\otimes} \varphi$ if and only if $M_s \models_{K_?} \varphi'$.*

Proof. [Sketch] Given Corollary 6.4 we can replace all occurrences of $[\mathsf{M}_\mathsf{s}]\psi$ within φ with an equivalent $[\alpha]\psi$ where $\alpha \in \mathcal{L}_?{}^{\text{act}}$. □

6.2 $\mathcal{K}45$

As in the previous subsection we give a lemma to simplify the construction that we will use, although as the definition of $\tau(?\varphi)$ is the same between $\mathcal{K}$ and $\mathcal{K}45$ we simply reuse Lemma 6.1 from the previous subsection.

Lemma 6.6 *Let $a \in A$, $\alpha \in \mathcal{L}_?{}^{act}$ where $\tau(\alpha) = \mathsf{M}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$, and $\mathsf{M}_\mathsf{s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{AM}$ such that $\mathsf{s}\mathsf{R}_a = \{\mathsf{t}\}$ for some $\mathsf{t} \in \mathsf{S}$ and $\mathsf{t}\mathsf{R}_a = \{\mathsf{t}\}$ Then let $\mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}') \in \mathcal{AM}$ where:*

$$
\begin{aligned}
\mathsf{S}' &= \mathsf{S} \cup \mathsf{S}^\alpha \cup \{\bar{\mathsf{t}}^\alpha \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \cup \{\mathsf{s}'\} \\
\mathsf{R}'_a &= \mathsf{R}_a \cup \mathsf{R}^\alpha_a \cup \{(\mathsf{s}', \bar{\mathsf{t}}^\alpha) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \cup \{(\bar{\mathsf{t}}^\alpha, \bar{\mathsf{u}}^\alpha) \mid \mathsf{t}^\alpha, \mathsf{u}^\alpha \in \mathsf{T}^\alpha\} \\
\mathsf{R}'_b &= \mathsf{R}_b \cup \mathsf{R}^\alpha_b \cup \{(\mathsf{s}', \mathsf{t}) \mid \mathsf{t} \in \mathsf{s}\mathsf{R}_b\} \text{ for } b \in A \setminus \{a\} \\
\mathsf{pre}' &= \mathsf{pre} \cup \{(\bar{\mathsf{t}}^\alpha, \mathsf{pre}^\alpha(\mathsf{t}^\alpha)) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\} \cup \{(\mathsf{s}', \mathsf{pre}(\mathsf{s}))\}
\end{aligned}
$$

Then $\tau(L_a\alpha) \otimes \mathsf{M}_\mathsf{s} \leftrightarrows \mathsf{M}'_{\mathsf{s}'}$.

Proposition 6.7 *Let $\mathsf{M}_\mathsf{s} \in \mathcal{AM}_{\mathcal{K}45}$ and let $n \in \mathbb{N}$. Then there exists $\alpha \in \mathcal{L}_?{}^{act}$ such that $\mathsf{M}_\mathsf{s} \leftrightarrows_n \tau(\alpha)$.*

Proof. By induction on n.

Suppose that $n = 0$. Let $\alpha = ?\mathsf{pre}(\mathsf{s})$ and $\tau(\alpha) = \mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}')$. From Definition 4.8 we have that $\mathsf{pre}(\mathsf{s}) = \mathsf{pre}'(\mathsf{s}')$, so $(\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'})$ satisfies **atoms** and therefore $\mathsf{M}_\mathsf{s} \leftrightarrows_0 \mathsf{M}'_{\mathsf{s}'}$.

Suppose that $n > 0$. By the induction hypothesis, for every $a \in A$, $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ there exists $\alpha^{a,\mathsf{t}} \in \mathcal{L}_?{}^{\text{act}}$ such that $\mathsf{M}_\mathsf{t} \leftrightarrows_{(n-1)} \tau(\alpha^{a,\mathsf{t}})$. For every $a \in A$, $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ let $\tau(\alpha^{a,\mathsf{t}}) = \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}} = ((\mathsf{S}^{a,\mathsf{t}}, \mathsf{R}^{a,\mathsf{t}}, \mathsf{pre}^{a,\mathsf{t}}), \mathsf{s}^{a,\mathsf{t}})$.

Let $\alpha = ?\mathsf{pre}(\mathsf{s}) \otimes \bigotimes_{a \in A} L_a(\bigsqcup_{\mathsf{t} \in \mathsf{s}\mathsf{R}_a} \alpha^{a,\mathsf{t}})$. Then from Lemmas 6.1 and 6.6: $\tau(\alpha) \leftrightarrows \mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}')$ where:

$$\mathsf{S}' = \bigcup_{a\in A, \mathsf{t}\in \mathsf{sR}_a} (\mathsf{S}^{a,\mathsf{t}}) \cup \{\bar{\mathsf{s}}^{a,\mathsf{t}} \mid a \in A, \mathsf{t} \in \mathsf{sR}_a\} \cup \{\mathsf{s}'\}$$

$$\mathsf{R}'_a = \bigcup_{b\in A, \mathsf{t}\in \mathsf{sR}_b} (\mathsf{R}^{b,\mathsf{t}}_a) \cup \{(\mathsf{s}', \bar{\mathsf{s}}^{a,\mathsf{t}}) \mid \mathsf{t} \in \mathsf{sR}_a\} \cup \{(\bar{\mathsf{s}}^{a,\mathsf{t}}, \bar{\mathsf{s}}^{a,\mathsf{u}}) \mid \mathsf{t}, \mathsf{u} \in \mathsf{sR}_a\} \cup$$
$$\{(\bar{\mathsf{s}}^{b,\mathsf{t}}, \mathsf{u}) \mid b \in A \setminus \{a\}, \mathsf{t} \in \mathsf{sR}_b, \mathsf{u} \in \mathsf{s}^{b,\mathsf{t}}\mathsf{R}^{b,\mathsf{t}}_a\} \text{ for } a \in A$$

$$\mathsf{pre}' = \bigcup_{a\in A, \mathsf{t}\in \mathsf{sR}_a} (\mathsf{pre}^{a,\mathsf{t}}) \cup \{(\bar{\mathsf{s}}^{a,\mathsf{t}}, \mathsf{pre}^{a,\mathsf{t}}(\mathsf{s}^{a,\mathsf{t}})) \mid a \in A, \mathsf{t} \in \mathsf{sR}_a\} \cup \{(\mathsf{s}', \mathsf{pre}(\mathsf{s}))\}$$

As in the proof of Proposition 6.3, we note for every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$ that $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarroweq \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}}$.

We need to show that $(\mathsf{M}_\mathsf{s}, \mathsf{M}'_{\mathsf{s}'})$ satisfies **atoms**, **forth-n-a** and **back-n-a** for every $a \in A$. We use reasoning similar to the proof of Proposition 6.3, however noting that the successors of s' in M' are not the same as in the construction used previously. We claim that each $\bar{\mathsf{s}}^{a,\mathsf{t}}$ state is $(n-1)$-bisimilar to the corresponding $\mathsf{s}^{a,\mathsf{t}}$ state. We show this by showing for every $0 \leq i \leq n-1$, $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$ that $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}} \leftrightarroweq_i \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$. We proceed by induction on i.

atoms By construction $\mathsf{pre}'(\bar{\mathsf{s}}^{a,\mathsf{t}}) = \mathsf{pre}'(\mathsf{s}^{a,\mathsf{t}})$.

forth-i-b Suppose that $0 < i \leq n-1$. Let $\mathsf{u} \in \bar{\mathsf{s}}^{a,\mathsf{t}}\mathsf{R}'_b$.

Suppose that $b = a$. By construction there exists $\mathsf{v} \in \mathsf{sR}_a$ such that $\mathsf{u} = \bar{\mathsf{s}}^{a,\mathsf{v}}$. From above $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarroweq \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}}$ and $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{v}}} \leftrightarroweq \mathsf{M}^{a,\mathsf{v}}_{\mathsf{s}^{a,\mathsf{v}}}$. By the outer induction hypothesis $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarroweq_{(n-1)} \mathsf{M}_\mathsf{t}$ and $\mathsf{M}^{a,\mathsf{v}}_{\mathsf{s}^{a,\mathsf{v}}} \leftrightarroweq_{(n-1)} \mathsf{M}_\mathsf{v}$. By transitivity $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarroweq_{(n-1)} \mathsf{M}_\mathsf{t}$ and $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{v}}} \leftrightarroweq_{(n-1)} \mathsf{M}_\mathsf{v}$. As $\mathsf{v} \in \mathsf{tR}_a$ from **back-$(n-1)$-a** there exists $\mathsf{w} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_a$ such that $\mathsf{M}'_\mathsf{w} \leftrightarroweq_{(n-2)} \mathsf{M}_\mathsf{v}$. By transitivity $\mathsf{M}'_\mathsf{w} \leftrightarroweq_{(n-2)} \mathsf{M}'_{\mathsf{s}^{a,\mathsf{v}}}$. By the induction hypothesis $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{v}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\mathsf{s}^{a,\mathsf{v}}}$. Therefore by transitivity $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{v}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_\mathsf{w}$.

Suppose that $b \neq a$. By construction $\bar{\mathsf{s}}^{a,\mathsf{t}}\mathsf{R}'_b = \mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_b$, so $\mathsf{u} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_b$ and we trivially have that $\mathsf{M}'_\mathsf{u} \leftrightarroweq \mathsf{M}'_\mathsf{u}$.

back-i-b Follows similar reasoning to **forth-i-b**.

Therefore for every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$ we have that $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}} \leftrightarroweq_{(n-1)} \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$.

We can now show that $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \mathsf{M}'_{\mathsf{s}'}$ by using the same reasoning as the proof for Proposition 6.3, using the $(n-1)$-bisimilar $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}}$ states in place of corresponding $\mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$ states.

Therefore $\mathsf{M}_\mathsf{s} \leftrightarroweq_n \tau(\alpha)$. □

Corollary 6.8 *Let* $\mathsf{M}_\mathsf{s} \in \mathcal{AM}_{\mathcal{K}45}$. *Then for every* $\varphi \in \mathcal{L}_\otimes$ *there exists* $\alpha \in \mathcal{L}_?{}^{act}$ *such that* $\vDash_{K45_\otimes} [\mathsf{M}_\mathsf{s}]\varphi \leftrightarrow [\tau(\alpha)]\varphi$.

Corollary 6.9 *Let* $\varphi \in \mathcal{L}_\otimes$. *Then there exists* $\varphi' \in \mathcal{L}_?$ *such that for every* $M_s \in \mathcal{K}45$*:* $M_s \vDash_{K45_\otimes} \varphi$ *if and only if* $M_s \vDash_{K45_?} \varphi'$.

6.3 $\mathcal{S}5$

Once more we give two lemmas to simplify the construction that we will use.

Lemma 6.10 *Let* $\varphi \in \mathcal{L}_?$ *and* $\mathsf{M}_\mathsf{s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{AM}$. *Then let* $\mathsf{M}'_\mathsf{s} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}) \in \mathcal{AM}$ *where:*

$$\mathsf{S}' = \mathsf{S}$$
$$\mathsf{R}'_a = \mathsf{R}_a \textit{ for } a \in A$$
$$\mathsf{pre}' = \mathsf{pre} \setminus \{(\mathsf{s}, \mathsf{pre}(\mathsf{s}))\} \cup \{(\mathsf{s}, \varphi \wedge \mathsf{pre}(\mathsf{s}))\}$$

Then $\tau(?\varphi) \otimes \mathsf{M_s} \leftrightarrows \mathsf{M}'_{\mathsf{s}'}$.

Lemma 6.11 *Let* $a \in A$*,* $\alpha \in \mathcal{L}_?{}^{act}$ *where* $\tau(\alpha) = \mathsf{M}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha)$*, and* $\mathsf{M_s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s}) \in \mathcal{AM}$ *such that* $\mathsf{sR}_a = \{\mathsf{s}\}$ *and* $\mathsf{pre}(\mathsf{s}) = \top$*. Then let* $\mathsf{M}'_\mathsf{s} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}) \in \mathcal{AM}$ *where:*

$$\mathsf{S}' = \mathsf{S} \cup \mathsf{S}^\alpha \cup \{\bar{\mathsf{t}}^\alpha \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\}$$
$$\mathsf{R}'_a = \mathsf{R}_a \cup \mathsf{R}^\alpha_a \cup (\{\mathsf{s}\} \cup \{\bar{\mathsf{t}}^\alpha \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\})^2$$
$$\mathsf{R}'_b = \mathsf{R}_b \cup \mathsf{R}^\alpha_b \cup (\{\bar{\mathsf{t}}^\alpha\} \cup \mathsf{t}^\alpha \mathsf{R}^\alpha_b)^2 \textit{ for } b \in A \setminus \{a\}$$
$$\mathsf{pre}' = \mathsf{pre} \cup \{(\bar{\mathsf{t}}^\alpha, \mathsf{pre}^\alpha(\mathsf{t}^\alpha)) \mid \mathsf{t}^\alpha \in \mathsf{T}^\alpha\}$$

Then $\tau(L_a(?\top, \alpha)) \otimes \mathsf{M_s} \leftrightarrows \mathsf{M}'_{\mathsf{s}'}$.

Proposition 6.12 *Let* $\mathsf{M_s} \in \mathcal{AM}_{\mathcal{S}5}$ *and let* $n \in \mathbb{N}$*. Then there exists* $\alpha \in \mathcal{L}_?{}^{act}$ *such that* $\mathsf{M_s} \leftrightarrows_n \tau(\alpha)$*.*

Proof. By induction on n.

Suppose that $n = 0$. Let $\alpha = ?\mathsf{pre}(\mathsf{s})$ and $\tau(\alpha) = \mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}')$. From Definition 4.12 we have that $\mathsf{pre}(\mathsf{s}) = \mathsf{pre}'(\mathsf{s}')$, so $(\mathsf{M_s}, \mathsf{M}'_{\mathsf{s}'})$ satisfies **atoms** and therefore $\mathsf{M_s} \leftrightarrows_0 \mathsf{M}'_{\mathsf{s}'}$.

Suppose that $n > 0$. By the induction hypothesis, for every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$ there exists $\alpha^{a,\mathsf{t}} \in \mathcal{L}_?{}^{\mathrm{act}}$ such that $\mathsf{M_t} \leftrightarrows_{(n-1)} \tau(\alpha^{a,\mathsf{t}})$. For every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$ let $\tau(\alpha^{a,\mathsf{t}}) = \mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}} = ((\mathsf{S}^{a,\mathsf{t}}, \mathsf{R}^{a,\mathsf{t}}, \mathsf{pre}^{a,\mathsf{t}}), \mathsf{s}^{a,\mathsf{t}})$.

Let $\alpha = ?\mathsf{pre}(\mathsf{s}) \otimes \bigotimes_{a \in A} L_a(?\top, \bigsqcup_{\mathsf{t} \in \mathsf{sR}_a} \alpha^{a,\mathsf{t}})$. Then from Lemmas 6.1 and 6.6: $\tau(\alpha) = \mathsf{M}'_{\mathsf{s}'} = ((\mathsf{S}', \mathsf{R}', \mathsf{pre}'), \mathsf{s}')$ where:

$$\mathsf{S}' = \bigcup_{a \in A, \mathsf{t} \in \mathsf{sR}_a} (\mathsf{S}^{a,\mathsf{t}}) \cup \{\bar{\mathsf{s}}^{a,\mathsf{t}} \mid a \in A, \mathsf{t} \in \mathsf{sR}_a\} \cup \{\mathsf{s}'\}$$
$$\mathsf{R}'_a = \bigcup_{b \in A, \mathsf{t} \in \mathsf{sR}_b} (\mathsf{R}^{b,\mathsf{t}}_a) \cup (\{\mathsf{s}'\} \cup \{\bar{\mathsf{s}}^{a,\mathsf{t}} \mid \mathsf{t} \in \mathsf{sR}_a\})^2 \cup \bigcup_{b \in A \setminus \{a\}, \mathsf{t} \in \mathsf{R}_b} (\{\bar{\mathsf{s}}^{b,\mathsf{t}}\} \cup \mathsf{s}^{b,\mathsf{t}} \mathsf{R}^{b,\mathsf{t}}_a)^2 \text{ for } a \in A$$
$$\mathsf{pre}' = \bigcup_{a \in A, \mathsf{t} \in \mathsf{sR}_a} (\mathsf{pre}^{a,\mathsf{t}}) \cup \{(\bar{\mathsf{s}}^{a,\mathsf{t}}, \mathsf{pre}^{a,\mathsf{t}}(\mathsf{s}^{a,\mathsf{t}})) \mid a \in A, \mathsf{t} \in \mathsf{sR}_a\} \cup \{(\mathsf{s}', \mathsf{pre}(\mathsf{s}))\}$$

We note that unlike the constructions used for Proposition 6.3 and Proposition 6.7, this construction does not have $\mathsf{M}'_\mathsf{u} \leftrightarrows \mathsf{M}^{a,\mathsf{t}}_\mathsf{u}$, as we do not have that $\mathsf{s}^{a,\mathsf{t}} \mathsf{R}'_a = \mathsf{s}^{a,\mathsf{t}} \mathsf{R}^{a,\mathsf{t}}_a$. Similar to the proof of Proposition 6.7 we claim that each $\bar{\mathsf{s}}^{a,\mathsf{t}}$ state is $(n-1)$-bisimilar to the corresponding s^t state. However in lieu of bisimilarity of $\mathsf{S}^{a,\mathsf{t}}$ states we need another result for these states. We also need to consider the additional state s', which due to reflexivity is also a successor of itself.

We need to show for every $0 \leq i \leq n-1$:

(i) For every $a \in A$: $\mathsf{M}'_{\mathsf{s}'} \leftrightarroweq_i \mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{s}}}$.

(ii) For every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$: $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}} \leftrightarroweq_i \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$.

(iii) For every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$, $\mathsf{u} \in \mathsf{S}^{a,\mathsf{t}}$, $\mathsf{v} \in \mathsf{S}$: if $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{u}} \leftrightarroweq_i \mathsf{M}_{\mathsf{v}}$ then $\mathsf{M}'_{\mathsf{u}} \leftrightarroweq_i \mathsf{M}_{\mathsf{v}}$.

We proceed by induction on i.

(i) For every $a \in A$: $\mathsf{M}'_{\mathsf{s}'} \leftrightarroweq_i \mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{s}}}$.

atoms By the outer induction hypothesis $\mathsf{M}^{a,\mathsf{s}}_{\mathsf{s}^{a,\mathsf{s}}} \leftrightarroweq_{(n-1)} \mathsf{M}_{\mathsf{s}}$ and so $\vdash \mathsf{pre}^{a,\mathsf{s}}(\mathsf{s}^{a,\mathsf{s}}) \leftrightarrow \mathsf{pre}(\mathsf{s})$. By construction $\mathsf{pre}'(\mathsf{s}') = \mathsf{pre}(\mathsf{s})$ and $\mathsf{pre}'(\bar{\mathsf{s}}^{a,\mathsf{s}}) = \mathsf{pre}^{a,\mathsf{s}}(\mathsf{s}^{a,\mathsf{s}})$ and therefore $\vdash \mathsf{pre}'(\mathsf{s}') \leftrightarrow \mathsf{pre}^{a,\mathsf{s}}(\bar{\mathsf{s}}^{a,\mathsf{s}})$.

forth-i-b Suppose that $0 < i \leq n-1$. Let $\mathsf{u} \in \mathsf{s}'\mathsf{R}'_a$.

Suppose that $b = a$. By construction $\mathsf{s}'\mathsf{R}'_a = \bar{\mathsf{s}}^{a,\mathsf{s}}\mathsf{R}'_a$ and we trivially have that $\mathsf{M}'_{\mathsf{u}} \leftrightarroweq \mathsf{M}'_{\mathsf{u}}$.

Suppose that $b \neq a$. By construction $\mathsf{s}'\mathsf{R}'_b = \{\bar{\mathsf{s}}^{b,\mathsf{t}} \mid \mathsf{t} \in \mathsf{sR}_b\} \cup \{\mathsf{s}'\}$ and $\bar{\mathsf{s}}^{a,\mathsf{s}}\mathsf{R}'_b = \mathsf{s}^{a,\mathsf{s}}\mathsf{R}^{a,\mathsf{s}}_b \cup \{\bar{\mathsf{s}}^{a,\mathsf{s}}\}$. Suppose that $\mathsf{u} = \mathsf{s}'$. Then by the induction hypothesis $\mathsf{M}'_{\mathsf{s}'} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{s}}}$. Suppose that $\mathsf{u} \in \{\bar{\mathsf{s}}^{b,\mathsf{t}} \mid \mathsf{t} \in \mathsf{sR}_b\}$. Then there exists $\mathsf{t} \in \mathsf{sR}_b$ such that $\mathsf{u} = \bar{\mathsf{s}}^{b,\mathsf{t}}$. By the outer induction hypothesis $\mathsf{M}^{a,\mathsf{s}}_{\mathsf{s}^{a,\mathsf{s}}} \leftrightarroweq_{(n-1)} \mathsf{M}_{\mathsf{s}}$. As $\mathsf{t} \in \mathsf{sR}_b$ then by **back-**$(n-1)$**-**b there exists $\mathsf{v} \in \mathsf{s}^{a,\mathsf{s}}\mathsf{R}^{a,\mathsf{s}}_b \subseteq \bar{\mathsf{s}}^{a,\mathsf{s}}\mathsf{R}'_b$ such that $\mathsf{M}^{a,\mathsf{s}}_{\mathsf{v}} \leftrightarroweq_{(n-2)} \mathsf{M}_{\mathsf{t}}$. Then by the inner induction hypothesis this implies $\mathsf{M}'_{\mathsf{v}} \leftrightarroweq_{(i-1)} \mathsf{M}_{\mathsf{t}}$. By the inner induction hypothesis $\mathsf{M}'_{\bar{\mathsf{s}}^{b,\mathsf{t}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\mathsf{s}^{b,\mathsf{t}}} \leftrightarroweq_{(i-1)} \mathsf{M}^{b,\mathsf{t}}_{\mathsf{s}^{b,\mathsf{t}}}$ and by the outer induction hypothesis $\mathsf{M}^{b,\mathsf{t}}_{\mathsf{s}^{b,\mathsf{t}}} \leftrightarroweq_{(n-1)} \mathsf{M}_{\mathsf{t}}$ so by transitivity $\mathsf{M}'_{\bar{\mathsf{s}}^{b,\mathsf{t}}} \leftrightarroweq_{(i-1)} \mathsf{M}_{\mathsf{t}}$. Therefore by transitivity we have that $\mathsf{M}'_{\bar{\mathsf{s}}^{b,\mathsf{t}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\mathsf{v}}$.

back-i-b Follows similar reasoning to **forth-i-b**.

(ii) For every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$: $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}} \leftrightarroweq_i \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$.

atoms By construction $\mathsf{pre}'(\bar{\mathsf{s}}^{a,\mathsf{t}}) = \mathsf{pre}'(\mathsf{s}^{a,\mathsf{t}})$.

forth-i-b Suppose that $0 < i \leq n-1$. Let $\mathsf{u} \in \bar{\mathsf{s}}^{a,\mathsf{t}}\mathsf{R}'_a$.

Suppose that $b = a$. By construction $\bar{\mathsf{s}}^{a,\mathsf{t}}\mathsf{R}'_a = \{\bar{\mathsf{s}}^{a,\mathsf{v}} \mid \mathsf{v} \in \mathsf{tR}_a\} \cup \{\mathsf{s}'\}$. Suppose that $\mathsf{u} \in \{\bar{\mathsf{s}}^{a,\mathsf{v}} \mid \mathsf{v} \in \mathsf{tR}_a\}$. Then there exists $\mathsf{v} \in \mathsf{tR}_a$ such that $\mathsf{u} = \bar{\mathsf{s}}^{a,\mathsf{v}}$. By the outer induction hypothesis $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{s}^{a,\mathsf{t}}} \leftrightarroweq_{(n-1)} \mathsf{M}_{\mathsf{t}}$. As $\mathsf{v} \in \mathsf{tR}_a$ then by **back-**$(n-1)$**-**a there exists $\mathsf{w} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}^{a,\mathsf{t}}_a \subseteq \mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_a$ such that $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{w}} \leftrightarroweq_{(n-2)} \mathsf{M}_{\mathsf{v}}$. Then by the inner induction hypothesis this implies $\mathsf{M}'_{\mathsf{w}} \leftrightarroweq_{(i-1)} \mathsf{M}_{\mathsf{v}}$. By the inner and outer induction hypothesis $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{v}}} \leftrightarroweq_{(i-1)} \mathsf{M}_{\mathsf{v}}$. Therefore by transitivity we have that $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{v}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\mathsf{w}}$. Suppose that $\mathsf{u} = \mathsf{s}'$. Then from the inner induction hypothesis $\mathsf{M}'_{\mathsf{s}'} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{s}}}$ and we can proceed using the same reasoning as in the case where $\mathsf{u} = \bar{\mathsf{s}}^{a,\mathsf{s}} \in \{\bar{\mathsf{s}}^{a,\mathsf{v}} \mid \mathsf{v} \in \mathsf{tR}_a\}$.

Suppose that $b \neq a$. By construction $\bar{\mathsf{s}}^{b,\mathsf{t}}\mathsf{R}'_b = \mathsf{s}^{a,\mathsf{t}}\mathsf{R}^{a,\mathsf{t}}_b \cup \{\bar{\mathsf{s}}^{b,\mathsf{t}}\}$. Suppose that $\mathsf{u} = \bar{\mathsf{s}}^{b,\mathsf{t}}$. By construction $\mathsf{s}^{a,\mathsf{t}} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_b$ and by the induction hypothesis $\mathsf{M}'_{\bar{\mathsf{s}}^{b,\mathsf{t}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$. Suppose that $\mathsf{u} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}^{a,\mathsf{t}}_b \subseteq \mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_b$. Then we trivially have that $\mathsf{M}'_{\mathsf{u}} \leftrightarroweq \mathsf{M}'_{\mathsf{u}}$.

back-i-b Follows similar reasoning to **forth-i-b**.

(iii) For every $a \in A$, $\mathsf{t} \in \mathsf{sR}_a$, $\mathsf{u} \in \mathsf{S}^{a,\mathsf{t}}$, $\mathsf{v} \in \mathsf{S}$: if $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{u}} \leftrightarroweq_i \mathsf{M}_{\mathsf{v}}$ then $\mathsf{M}'_{\mathsf{u}} \leftrightarroweq_i \mathsf{M}_{\mathsf{v}}$.

Suppose that $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{u}} \leftrightarroweq_i \mathsf{M}_{\mathsf{v}}$.

atoms As $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{u}} \leftrightarroweq_i \mathsf{M}_{\mathsf{v}}$ then $\vdash \mathsf{pre}^{a,\mathsf{t}}(\mathsf{u}) \leftrightarrow \mathsf{pre}(\mathsf{v})$. By construction $\mathsf{pre}'(\mathsf{u}) = \mathsf{pre}^{a,\mathsf{t}}(\mathsf{u})$ and therefore $\vdash \mathsf{pre}'(\mathsf{u}) \leftrightarrow \mathsf{pre}(\mathsf{v})$.

forth-i-b Suppose that $0 < i \leq n-1$. Let $\mathsf{w} \in \mathsf{u}\mathsf{R}'_b$.

Suppose that $\mathsf{u} \neq \mathsf{s}^{a,\mathsf{t}}$ or $b = a$. By construction $\mathsf{u}\mathsf{R}'_a = \mathsf{u}\mathsf{R}^{a,\mathsf{t}}_a$ and so $\mathsf{w} \in \mathsf{u}\mathsf{R}^{a,\mathsf{t}}_a$. As $\mathsf{w} \in \mathsf{u}\mathsf{R}^{a,\mathsf{t}}_a$ then by **forth-i-b** there exists $\mathsf{x} \in \mathsf{v}\mathsf{R}_b$ such that $\mathsf{M}^{a,\mathsf{t}}_{\mathsf{w}} \leftrightarroweq_{(i-1)} \mathsf{M}_{\mathsf{x}}$. By the induction hypothesis $\mathsf{M}'_{\mathsf{w}} \leftrightarroweq_{(i-1)} \mathsf{M}_{\mathsf{x}}$.

Suppose that $\mathsf{u} = \mathsf{s}^{a,\mathsf{t}}$ and $b \neq a$. By construction $\mathsf{s}^{a,\mathsf{t}}\mathsf{R}'_a = \mathsf{s}^{a,\mathsf{t}}\mathsf{R}^{a,\mathsf{t}}_a \cup \{\bar{\mathsf{s}}^{a,\mathsf{t}}\}$. Suppose that $\mathsf{w} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}^{a,\mathsf{t}}_a$. We proceed using the same reasoning as above, where $\mathsf{w} \in \mathsf{u}\mathsf{R}^{a,\mathsf{t}}_a$. Suppose that $\mathsf{w} = \bar{\mathsf{s}}^{a,\mathsf{t}}$. By the induction hypothesis $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}} \leftrightarroweq_{(i-1)} \mathsf{M}'_{\mathsf{s}^{a,\mathsf{t}}}$ and we proceed using the same reasoning above, where $\mathsf{w} = \mathsf{s}^{a,\mathsf{t}} \in \mathsf{s}^{a,\mathsf{t}}\mathsf{R}^{a,\mathsf{t}}_a$.

back-i-b Follows similar reasoning to **forth-i-b**.

Therefore for every $a \in A$, $\mathsf{t} \in \mathsf{s}\mathsf{R}_a$ we have that $\mathsf{M}'_{\mathsf{s}'} \leftrightarroweq_{(n-1)} \mathsf{M}_{\mathsf{s}}$ and $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}} \leftrightarroweq_{(n-1)} \mathsf{M}_{\mathsf{t}}$. We can now show that $\mathsf{M}_{\mathsf{s}'} \leftrightarroweq_n \mathsf{M}_{\mathsf{s}}$ by using the same reasoning as the proof for Proposition 6.3, using the $(n-1)$-bisimilar $\mathsf{M}'_{\bar{\mathsf{s}}^{a,\mathsf{t}}}$ in place of corresponding $\mathsf{M}'_{\mathsf{s}^{\mathsf{t}}}$ states. □

Corollary 6.13 *Let $\mathsf{M}_{\mathsf{s}} \in \mathcal{AM}_{\mathcal{S}5}$. Then for every $\varphi \in \mathcal{L}_\otimes$ there exists $\alpha \in \mathcal{L}_?{}^{act}$ such that $\models_{\mathcal{S}5_\otimes} [\mathsf{M}_{\mathsf{s}}]\varphi \leftrightarrow [\tau(\alpha)]\varphi$.*

Corollary 6.14 *Let $\varphi \in \mathcal{L}_\otimes$. Then there exists $\varphi' \in \mathcal{L}_?$ such that for every $M_s \in \mathcal{S}5$: $M_s \models_{\mathcal{S}5_\otimes} \varphi$ if and only if $M_s \models_{\mathcal{S}5_?} \varphi'$.*

7 Synthesis

In the following subsections we give a computational method for synthesising action formulae to achieve epistemic goals, whenever those goals are achievable. We note that the notion of when an epistemic goal is achievable is captured by the refinement quantifiers of refinement modal logic [14,10], which are also included in the arbitrary action formula logic, and so in this section we will refer to the full arbitrary action formula logic, keeping in mind the correspondence with arbitrary action model logic mentioned in Section 4.

7.1 $\mathcal{K}$

Proposition 7.1 *For every $\varphi \in \mathcal{L}_?$ there exists $\alpha \in \mathcal{L}_?{}^{act}$ such that $\vdash [\alpha]\varphi$ and $\vdash \exists\varphi \to \langle\alpha\rangle\varphi$.*

Proof. Without loss of generality we assume that φ is in disjunctive normal form. We proceed by induction on the structure of φ.

Suppose that $\varphi = \psi \vee \chi$. By the induction hypothesis there exists $\alpha^\psi, \alpha^\chi \in \mathcal{L}_?{}^{\mathrm{act}}$ such that $\vdash [\alpha^\psi]\psi$, $\vdash \exists\psi \to \langle\alpha^\psi\rangle\psi$, $\vdash [\alpha^\chi]\chi$ and $\vdash \exists\chi \to \langle\alpha^\chi\rangle\chi$. Let $\alpha = \alpha^\psi \sqcup \alpha^\chi$. Then:

$$\vdash [\alpha^\psi](\psi \vee \chi) \wedge [\alpha^\chi](\psi \vee \chi) \tag{9}$$

$$\vdash [\alpha^\psi \sqcup \alpha^\chi](\psi \vee \chi) \tag{10}$$

(9) follows from the induction hypothesis and (10) follows from **LU**.

Further:

$$\vdash (\exists\psi \vee \exists\chi) \rightarrow (\langle\alpha^{\psi}\rangle(\psi \vee \chi) \vee \langle\alpha^{\chi}\rangle(\psi \vee \chi)) \tag{11}$$

$$\vdash (\exists\psi \vee \exists\chi) \rightarrow \langle\alpha^{\psi} \sqcup \alpha^{\chi}\rangle(\psi \vee \chi) \tag{12}$$

$$\vdash \exists(\psi \vee \chi) \rightarrow \langle\alpha^{\psi} \sqcup \alpha^{\chi}\rangle(\psi \vee \chi) \tag{13}$$

(11) follows from the induction hypothesis, (12) follows from **LU** and (13) follows from **R**.

Suppose that $\varphi = \pi \wedge \bigwedge_{b \in B \subseteq A} \nabla_b \Gamma_b$. By the induction hypothesis for every $b \in B$, $\gamma \in \Gamma_b$ there exists $\alpha^{\gamma} \in \mathcal{L}_?^{\text{act}}$ such that $\vdash [\alpha^{\gamma}]\gamma$ and $\vdash \exists\gamma \rightarrow \langle\alpha^{\gamma}\rangle\gamma$. Let $\alpha = ?\exists\varphi \otimes \bigotimes_{b \in B} L_b(\bigsqcup_{\gamma \in \Gamma_b} \alpha^{\gamma})$.

Then for every $b \in B$:

$$\vdash [\bigsqcup_{\gamma \in \Gamma_b} \alpha^{\gamma}] \bigvee_{\gamma \in \Gamma} \gamma \tag{14}$$

$$\vdash \Box_b [\bigsqcup_{\gamma \in \Gamma_b} \alpha^{\gamma}] \bigvee_{\gamma \in \Gamma} \gamma \tag{15}$$

$$\vdash [L_b(\bigsqcup_{\gamma \in \Gamma_b} \alpha^{\gamma})]\Box_b \bigvee_{\gamma \in \Gamma} \gamma \tag{16}$$

$$\vdash [\bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})]\Box_b \bigvee_{\gamma \in \Gamma} \gamma \tag{17}$$

$$\vdash [?\exists\varphi][\bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})]\Box_b \bigvee_{\gamma \in \Gamma} \gamma \tag{18}$$

$$\vdash [?\exists\varphi \otimes \bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})]\Box_b \bigvee_{\gamma \in \Gamma} \gamma \tag{19}$$

(14) follows from the induction hypothesis and **LU**, (15) follows from **NecK**, (16) follows from **LK1**, (17) follows from **LK2** and **LS**, (18) follows from **NecL** and (19) follows from **LS**.

Further:

$$\vdash \exists\varphi \rightarrow \bigwedge_{b \in B, \gamma \in \Gamma_b} \Diamond_b \exists\gamma \tag{20}$$

$$\vdash \exists\varphi \rightarrow \bigwedge_{b \in B, \gamma \in \Gamma_b} \Diamond_b \langle\alpha^{\gamma}\rangle\gamma \tag{21}$$

$$\vdash \exists\varphi \rightarrow \bigwedge_{b \in B, \gamma \in \Gamma_b} \Diamond_b \langle \bigsqcup_{\gamma' \in \Gamma_b} \alpha^{\gamma'}\rangle\gamma \tag{22}$$

$$\vdash \exists\varphi \rightarrow \langle\bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})\rangle \bigwedge_{b \in B, \gamma \in \Gamma_b} \Diamond_b \gamma \tag{23}$$

$$\vdash \exists\varphi \rightarrow \langle ?\exists\varphi \otimes \bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})\rangle \bigwedge_{b \in B, \gamma \in \Gamma_b} \Diamond_b \gamma \tag{24}$$

$$\vdash [?\exists\varphi \otimes \bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})] \bigwedge_{b \in B, \gamma \in \Gamma_b} \Diamond_b \gamma \tag{25}$$

$$\vdash [?\exists\varphi \otimes \bigotimes_{c \in B} L_c(\bigsqcup_{\gamma \in \Gamma_c} \alpha^{\gamma})](\pi \wedge \bigwedge_{b \in B} \nabla_b \Gamma_b) \tag{26}$$

(20) follows from **RK**, (21) follows from the induction hypothesis, (22) follows from **LU**, (23) follows from **LK1**, **LK2** and **LS**, (24) and (25) follow from **LT**, and (26) follows from (19), **RP LC** and the definition of the cover operator.

Therefore $\vdash [\alpha]\varphi$.

Finally:

$$\vdash \langle \bigotimes_{c\in B} L_c(\bigsqcup_{\gamma\in\Gamma_c} \alpha^\gamma)\rangle\top \leftrightarrow \top \tag{27}$$

$$\vdash \langle ?\exists\varphi \otimes \bigotimes_{c\in B} L_c(\bigsqcup_{\gamma\in\Gamma_c} \alpha^\gamma)\rangle\top \leftrightarrow \exists\varphi \tag{28}$$

$$\vdash \exists\varphi \rightarrow \langle\alpha\rangle\top \tag{29}$$

$$\vdash \exists\varphi \rightarrow \langle\alpha\rangle\varphi \tag{30}$$

(27) follows from **LS** and **LP**, (28) follows from **LS** and **LT**, (29) follows from (28), (30) follows from (26) and (29),

Therefore $\vdash \exists\varphi \rightarrow \langle\alpha\rangle\varphi$. □

Corollary 7.2 *For every $M_s \in \mathcal{K}$ and $\varphi \in \mathcal{L}_{\otimes\forall}$: $M_s \models \exists\varphi$ if and only if there exists $\mathsf{M_s} \in \mathcal{AM}$ such that $M_s \models \langle\mathsf{M_s}\rangle\varphi$.*

7.2 $\mathcal{K}45$

Proposition 7.3 *For every $\varphi \in \mathcal{L}_?$ there exists $\alpha \in \mathcal{L}_?{}^{act}$ such that $\vdash [\alpha]\varphi$ and $\vdash \exists\varphi \rightarrow \langle\alpha\rangle\varphi$.*

Proof. Without loss of generality we assume that φ is in alternating disjunctive normal form. We use the same reasoning as in the proof of Proposition 7.1, substituting $\mathbf{AFL_{K45}}$ axioms for the corresponding $\mathbf{AFL_K}$ axioms, noting that the alternating disjunctive normal form gives the $(A\setminus\{a\})$-restricted properties required for **LK1** and the $\mathbf{RML_{K45}}$ axioms **RK45**, **RComm** and **RDist** to be applicable. □

Corollary 7.4 *For every $M_s \in \mathcal{K}45$ and $\varphi \in \mathcal{L}_{\otimes\forall}$: $M_s \models \exists\varphi$ if and only if there exists $\mathsf{M_s} \in \mathcal{AM}_{\mathcal{K}45}$ such that $M_s \models \langle\mathsf{M_s}\rangle\varphi$.*

7.3 $\mathcal{S}5$

Proposition 7.5 *For every $\varphi \in \mathcal{L}_?$ there exists $\alpha \in \mathcal{L}_?{}^{act}$ such that $\vdash [\alpha]\varphi$ and $\vdash \exists\varphi \rightarrow \langle\alpha\rangle\varphi$.*

Proof. Without loss of generality, assume that φ is a disjunction of explicit formulae. We proceed by induction on the structure of φ.

Suppose that $\varphi = \psi \vee \chi$. We use the same reasoning as in the proof of Proposition 7.1.

Suppose that $\varphi = \pi \wedge \gamma^0 \wedge \bigwedge_{a\in A} \nabla_a\Gamma_a$ is an explicit formula. By the induction hypothesis for every $a \in A$, $\gamma \in \Gamma_a$ there exists $\alpha^{a,\gamma} \in \mathcal{L}_?{}^{\mathrm{act}}$ such that $\vdash [\alpha^{a,\gamma}]\gamma$ and $\vdash \exists\gamma \rightarrow \langle\alpha^{a,\gamma}\rangle\gamma$, where $\tau(\alpha^{a,\gamma}) = \mathsf{M}^{a,\gamma}_{\mathsf{s}^{a,\gamma}} = ((\mathsf{S}^{a,\gamma}, \mathsf{R}^{a,\gamma}, \mathsf{pre}^{a,\gamma}), \mathsf{s}^{a,\gamma})$.

Let $\alpha = ?\exists\gamma^0 \otimes \bigotimes_{a\in A} L_a(?\top, \bigsqcup_{\gamma\in\Gamma_a} \alpha^{a,\gamma})$. Then from Lemmas 6.10 and 6.11: $\tau(\alpha) \leftrightarrows \mathsf{M_s} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{s})$ where:

$$\mathsf{S} = \bigcup_{a\in A,\gamma\in\Gamma_a} \mathsf{S}^{a,\gamma} \cup \{\bar{\mathsf{s}}^{a,\gamma} \mid a \in A, \gamma \in \Gamma_a\} \cup \{\mathsf{s}\}$$

$$\mathsf{R}_a = \bigcup_{b\in A,\gamma\in\Gamma_b} \mathsf{R}_a^{b,\gamma} \cup (\{\mathsf{s}\} \cup \{\bar{\mathsf{s}}^{a,\gamma} \mid \gamma \in \Gamma_a\})^2 \cup \bigcup_{b\in A\setminus\{a\},\gamma\in\Gamma_b} (\{\bar{\mathsf{s}}^{b,\gamma}\} \cup \mathsf{s}^{b,\gamma}\mathsf{R}_a^{b,\gamma})^2 \text{ for } a \in A$$

$$\mathsf{pre} = \bigcup_{a\in A,\gamma\in\Gamma_a} \mathsf{pre}^{a,\gamma} \cup \{(\bar{\mathsf{s}}^{a,\gamma}, \mathsf{pre}^{a,\gamma}(\mathsf{s}^{a<,\gamma})) \mid a \in A, \gamma \in \gamma_a\} \cup \{(\mathsf{s}, \exists\gamma^0)\}$$

Let $\Psi = \{\psi \leq \gamma \mid a \in A, \gamma \in \Gamma_a\}$. We need to show for every $\psi \in \Psi$:

(i) For every $a \in A$: $\vdash [\mathsf{M}_\mathsf{s}]\psi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma^0}}]\psi$.

(ii) For every $a \in A$, $\gamma \in \Gamma_a$: $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\psi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\psi$.

(iii) For every $a \in A$, $\gamma \in \Gamma_a$, $\mathsf{u} \in \mathsf{S}^{a,\gamma}$: $\vdash [\mathsf{M}_\mathsf{u}]\psi \leftrightarrow [\mathsf{M}^{a,\gamma}_\mathsf{u}]\psi$.

We proceed by induction on ψ.

(i) For every $a \in A$: $\vdash [\mathsf{M}_\mathsf{s}]\psi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma^0}}]\psi$.
Suppose that $\psi = p$ where $p \in P$. This follows trivially from **AP**.
Suppose that $\psi = \neg\chi$ or that $\psi = \chi_1 \wedge \chi_2$. These cases follow trivially from the induction hypothesis.
Suppose that $\psi = \Box_a\chi$. By construction $\mathsf{s}\mathsf{R}_a = \bar{\mathsf{s}}^{a,\gamma^0}\mathsf{R}_a$ and $\mathsf{pre}(\mathsf{s}) = \mathsf{pre}(\bar{\mathsf{s}}^{a,\gamma^0})$ and so $\vdash [\mathsf{M}_\mathsf{s}]\Box_a\chi \leftrightarrow [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\Box_a\chi$ follows from **AK** trivially.
Suppose that $\psi = \Box_b\chi$ where $b \neq a$. By construction $\mathsf{s}\mathsf{R}_b = \{\mathsf{s}\} \cup \mathsf{s}^{b,\gamma^0}\mathsf{R}_b$ and $\bar{\mathsf{s}}^{a,\gamma^0}\mathsf{R}_b = \{\bar{\mathsf{s}}^{a,\gamma^0}\} \cup \mathsf{s}^{a,\gamma^0}\mathsf{R}_b^{a,\gamma^0}$. As φ is an explicit formula and $\Box_b\chi \in \Psi$ then either $\vdash \gamma^0 \to \Box_b\chi$ or $\vdash \gamma^0 \to \neg\Box_b\chi$. Suppose that $\vdash \gamma^0 \to \Box_b\chi$. Then for every $\gamma \in \Gamma_b$ we have $\vdash \gamma \to \Box_b\chi$. By the outer induction hypothesis $\vdash [\mathsf{M}^{b,\gamma}_{\mathsf{s}^{b,\gamma}}]\gamma$ and so $\vdash [\mathsf{M}^{b,\gamma}_{\mathsf{s}^{b,\gamma}}]\chi$. By the inner induction hypothesis $\vdash [\mathsf{M}_{\mathsf{s}^{b,\gamma}}]\chi$. As $\gamma^0 \in \Gamma_b$ then $\vdash [\mathsf{M}_{\mathsf{s}^{b,\gamma^0}}]\chi$ and so by the inner induction hypothesis $\vdash [\mathsf{M}_\mathsf{s}]\chi$. So $\vdash [\mathsf{M}_{\mathsf{s}\mathsf{R}_b}]\chi$ and therefore $\vdash [\mathsf{M}_\mathsf{s}]\Box_b\chi$ follows from **AK**. By the outer induction hypothesis $\vdash [\mathsf{M}^{a,\gamma^0}_{\mathsf{s}^{a,\gamma^0}}]\gamma^0$ and so $\vdash [\mathsf{M}^{a,\gamma^0}_{\mathsf{s}^{a,\gamma^0}}]\Box_b\chi$. From **AK** we have $\vdash \exists\gamma^0 \to \Box_b[\mathsf{M}^{a,\gamma^0}_{\mathsf{s}^{a,\gamma^0}\mathsf{R}_b^{a,\gamma^0}}]\chi$. By the inner induction hypothesis $\vdash [\mathsf{M}^{a,\gamma^0}_{\mathsf{s}^{a,\gamma^0}\mathsf{R}_b^{a,\gamma^0}}]\chi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma^0}\mathsf{R}_b^{a,\gamma^0}}]\chi$. and as $\vdash [\mathsf{M}_\mathsf{s}]\chi$ then $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\chi$. So we have $\vdash [\mathsf{M}^{a,\gamma^0}_{\mathsf{s}^{a,\gamma^0}\mathsf{R}_b^{a,\gamma^0}}]\chi \leftrightarrow [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}\mathsf{R}_b}]\chi$ and $\vdash \exists\gamma^0 \to \Box_b[\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}\mathsf{R}_b}]\chi$ and so $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\Box_b$ follows from **AK**. Therefore $\vdash [\mathsf{M}_\mathsf{s}]\Box_b\chi \leftrightarrow [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\Box_b\chi$. Suppose that $\vdash \gamma^0 \to \neg\Box_b\chi$. A dual argument can be used to show that $\vdash \neg[\mathsf{M}_\mathsf{s}]\Box_b\chi$ and $\vdash \neg[\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\Box_b\chi$ and therefore $\vdash [\mathsf{M}_\mathsf{s}]\Box_b\chi \leftrightarrow [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\Box_b\chi$.

(ii) For every $a \in A$, $\gamma \in \Gamma_a$: $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\psi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\psi$.
Suppose that $\psi = p$ where $p \in P$. This follows trivially from **AP**.
Suppose that $\psi = \neg\chi$ or that $\psi = \chi_1 \wedge \chi_2$. These cases follow trivially from the induction hypothesis.
Suppose that $\psi = \Box_a\chi$. By construction $\bar{\mathsf{s}}^{a,\gamma}\mathsf{R}_a = \{\mathsf{s}\} \cup \{\bar{\mathsf{s}}^{a,\gamma} \mid \delta \in \Gamma_a\}$ and $\mathsf{s}^{a,\gamma}\mathsf{R}_a = \mathsf{s}^{a,\gamma}\mathsf{R}_a^{a,\gamma}$. As φ is an explicit formula and $\Box_a\chi \in \Psi$ then either

$\vdash \gamma \to \Box_a\chi$ or $\vdash \gamma \to \neg\Box_a\chi$. Suppose that $\vdash \gamma \to \Box_a\chi$. Then for every $\delta \in \Gamma_a$ we have $\vdash \delta \to \Box_a\chi$. By the outer induction hypothesis $\vdash [\mathsf{M}^{a,\delta}_{\mathsf{s}^{a,\delta}}]\delta$ and so $\vdash [\mathsf{M}^{a,\delta}_{\mathsf{s}^{a,\delta}}]\chi$. By the inner induction hypothesis $\vdash [\mathsf{M}_{\mathsf{s}^{a,\delta}}]\chi$ and $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\delta}}]\chi$. As $\gamma^0 \in \Gamma_a$ then $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma^0}}]\chi$ and by the inner induction hypothesis $\vdash [\mathsf{M}_\mathsf{s}]\chi$. So $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}\mathsf{R}_a}]\chi$ and therefore $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\Box_a\chi$ follows from **AK**. By the outer induction hypothesis $\vdash [\mathsf{M}^{a,\gamma}_{\mathsf{s}^{a,\gamma}}]\gamma$ and so $\vdash [\mathsf{M}^{a,\gamma}_{\mathsf{s}^{a,\gamma}}]\Box_a\chi$. From **AK** we have $\vdash \exists\gamma \to \Box_a[\mathsf{M}^{a,\gamma}_{\mathsf{s}^{a,\gamma}\mathsf{R}^{a,\gamma}_a}]\chi$. By the inner induction hypothesis $\vdash [\mathsf{M}^{a,\gamma}_{\mathsf{s}^{a,\gamma}\mathsf{R}^{a,\gamma}_a}]\chi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma}\mathsf{R}^{a,\gamma}_a}]\chi$ so $\vdash \exists\gamma \to \Box_a[\mathsf{M}_{\mathsf{s}^{a,\gamma}\mathsf{R}_a}]\chi$ and so $\vdash [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\Box_a\chi$ follows from **AK**.

Suppose that $\vdash \gamma \to \neg\Box_b\chi$ where $b \neq a$. Therefore $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\Box_a\chi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\Box_a\chi$. A dual argument can be used to show that $\vdash \neg[\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\Box_a\chi$ and $\vdash \neg[\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\Box_a\chi$ and therefore $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\Box_a\chi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\Box_a\chi$.

Suppose that $\psi = \Box_b\chi$ where $b \neq a$. By construction $\bar{\mathsf{s}}^{a,\gamma}\mathsf{R}_b = \mathsf{s}^{a,\gamma}\mathsf{R}_b$ and $\mathsf{pre}(\bar{\mathsf{s}}^{a,\gamma}) = \mathsf{pre}(\mathsf{s}^{a,\gamma})$ and so $\vdash [\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}]\Box_b\chi \leftrightarrow [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\Box_b\chi$ follows from **AK** trivially.

(iii) For every $a \in A$, $\gamma \in \Gamma_a$, $\mathsf{u} \in \mathsf{S}^{a,\gamma}$: $\vdash [\mathsf{M}_\mathsf{u}]\psi \leftrightarrow [\mathsf{M}^{a,\gamma}_\mathsf{u}]\psi$.

Suppose that $\psi = p$ where $p \in P$. This follows trivially from **AP**.

Suppose that $\psi = \neg\chi$ or that $\psi = \chi_1 \wedge \chi_2$. These cases follow trivially from the induction hypothesis.

Suppose that $\psi = \Box_a\chi$. By construction $\mathsf{u}\mathsf{R}_a = \mathsf{u}\mathsf{R}^{a,\gamma}_a$ and $\mathsf{pre}(\mathsf{u}) = \mathsf{pre}^{a,\gamma}(\mathsf{u})$ and so $\vdash [\mathsf{M}_\mathsf{u}]\Box_a\chi \leftrightarrow [\mathsf{M}^{a,\gamma}_\mathsf{u}]\Box_a\chi$ follows from **AK** and the induction hypothesis trivially.

Suppose that $\psi = \Box_b\chi$ where $b \neq a$. By construction $\mathsf{u}\mathsf{R}_a = \mathsf{u}\mathsf{R}^{a,\gamma}_a$ or $\mathsf{u}\mathsf{R}_a = \{\bar{\mathsf{s}}^{a,\gamma}\} \cup \mathsf{u}\mathsf{R}^{a,\gamma}_a$ and $\mathsf{pre}(\mathsf{u}) = \mathsf{pre}^{a,\gamma}(\mathsf{u})$ and so $\vdash [\mathsf{M}_\mathsf{u}]\Box_b\chi \leftrightarrow [\mathsf{M}^{a,\gamma}_\mathsf{u}]\Box_b\chi$ follows from **AK** and the induction hypothesis trivially.

Therefore for every $a \in A$, $\gamma \in \Gamma_a$ we have that $\vdash [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\gamma$ and $\vdash [\mathsf{M}_\mathsf{s}]\gamma^0$. Therefore for every $a \in A$ we have $\vdash [\mathsf{M}_{\mathsf{s}\mathsf{R}_a}]\bigvee_{\gamma\in\Gamma_a}\gamma$ and so from **AK** we have that $\vdash [\mathsf{M}_\mathsf{s}]\Box_a\bigvee_{\gamma\in\Gamma_a}\gamma$.

As φ is an explicit formula, from **RDist**, **RS5** and **RComm** we have that $\exists\varphi \to \pi \wedge \bigwedge_{a\in A,\gamma\in\Gamma_a}\Diamond_a\exists\gamma$. By construction for every $a \in A$, $\gamma \in \Gamma_a$ we have $\mathsf{pre}(\bar{\mathsf{s}}^{a,\gamma}) = \exists\gamma$ and from above we have $\vdash [\mathsf{M}_{\mathsf{s}^{a,\gamma}}]\gamma$ therefore $\vdash \exists\varphi \to \pi \wedge \bigwedge_{a\in A,\gamma\in\Gamma_a}\Diamond_a\langle\mathsf{M}_{\bar{\mathsf{s}}^{a,\gamma}}\rangle\gamma$. Therefore by **AK** we have $\vdash \exists\varphi \to \langle\mathsf{M}_\mathsf{s}\rangle(\pi\wedge\bigwedge_{a\in A,\gamma\in\Gamma_a}\Diamond_a\gamma)$. From above we have $\vdash [\mathsf{M}_\mathsf{s}]\Box_a\bigvee_{\gamma\in\Gamma_a}\gamma$ and therefore $\vdash \exists\varphi \to [\mathsf{M}_\mathsf{s}]\varphi$. As $\vdash \varphi \to \gamma^0$ then $\vdash \exists\varphi \to \exists\gamma^0$ and so $\vdash \exists\varphi \to \langle\mathsf{M}_\mathsf{s}\rangle\varphi$.

Let $\alpha' = ?\exists\varphi \otimes \alpha$. By **LS** we have $\vdash [\alpha']\varphi \leftrightarrow [?\exists\varphi][\alpha]\varphi$. By **LT** we have $\vdash [\alpha']\varphi \leftrightarrow (\exists\varphi \to [\alpha]\varphi)$. From above we have $\vdash \exists\varphi \to [\alpha]\varphi$ and therefore $\vdash [\alpha']\varphi$. By **LS** we have $\vdash \langle\alpha'\rangle\varphi \leftrightarrow \langle?\exists\varphi\rangle\langle\alpha\rangle\varphi$. By **LT** we have $\vdash \langle\alpha'\rangle\varphi \leftrightarrow (\exists\varphi \wedge \langle\alpha\rangle\varphi)$. From above we have $\vdash \exists\varphi \to \langle\alpha\rangle\varphi$ and therefore $\vdash \exists\varphi \to \langle\alpha'\rangle\varphi$. □

Corollary 7.6 *For every $M_s \in \mathcal{S5}$ and $\varphi \in \mathcal{L}_{\otimes\forall}$: $M_s \models \exists\varphi$ if and only if there exists $\mathsf{M}_\mathsf{s} \in \mathcal{AM}_{\mathcal{S5}}$ such that $M_s \models \langle\mathsf{M}_\mathsf{s}\rangle\varphi$.*

8 Related work

Several other papers have addressed the problem of describing and reasoning about epistemic actions. One of the most important works in this area is the work of Baltag, Moss and Solecki [8] which introduced the notion of action model logic, building on the earlier work of Gerbrandy and Groeneveld [17]. In later work Baltag and Moss extended action model logic to consider epistemic programs [7] which are expressions built from action models using such operators as sequential composition, non-deterministic choice and iteration. The atoms of these programs are action models, so the approach is still inherently semantic in nature. The logic is unable to decompose the program beyond the level of the atoms, which themselves may be complex semantic objects.

The relational actions of van Ditmarsch [12] provides a syntactic mechanism for describing an epistemic action, and provides the foundation for a lot of the work presented in this paper. The relational actions are constructed using essentially the same operators as in the language of action formulae. While the language is very similar, the semantics given are quite different [16]. In the logic of epistemic actions the semantics are given in such a way that worlds in a model are specified with respect to subsets of agents, so that the model is restricted to agents for whom the epistemic action was applied. The semantics were also specific to $\mathcal{S5}$, and non-trivial to generalise to other epistemic logics. A version of relational actions with concurrency is able to describe any $\mathcal{S5}$ action model, although it is unknown whether the expressivity of concurrent relational actions is greater than that of action models [6]. Here we have generalised the approach and provided a correspondence theorem for action model logic. This has allowed us to retain the more familiar semantics of epistemic logic, generalise the logic to $\mathcal{K}$ and $\mathcal{K}45$ as well as access existing synthesis results for dynamic epistemic logic [19].

The synthesis result presented here is built on the work of Hales [19] which gave a method to build an action model to satisfy a given epistemic goal. This construction inspired the syntactic description of epistemic actions and approach that we have used in this paper.

Related synthesis results have been given by Aucher, et al. [2,3,4] which presents an event model language and uses it to give a thorough exploration of the relationship between epistemic models, action models and epistemic goals. Aucher defines a logic for action models and provides calculi to describe epistemic progression (what is true after executing a given action in a given model) epistemic regression (what is the most general precondition for an epistemic action given an epistemic goal) and epistemic planning (what action is sufficient to achieve an epistemic goal given some precondition). In future work we hope to extend the correspondence between action formula logic and action models to include Aucher's event model language.

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